THE MATHEMATICAL GAZETTE

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THE HAMILTONIAN REVIVAL.

BY E. T. WHITTAKER.

ONE of the most agreeable fruits of the renaissance which has followed the achievement of national independence in Ireland has been the decision to make accessible, in a form worthy of their importance, the writings of the greatest Irish mathematician and natural philosopher. The responsibility for the work was undertaken by the Royal Irish Academy, with the cooperation of the University of Dublin and the National University of Ireland, and with assistance from the Royal Society: and the editorship was entrusted to Professor A. W. Conway, F.R.S., of University College, Dublin, and Professor J. L. Synge of Trinity College, Dublin. The first volume, devoted to Hamilton's writings on Geometrical Optics, was published in 1931; and we have now to congratulate the editors (Professor Synge having meanwhile been succeeded by Professor A. J. McConnell) on the appearance of the second volume * which contains the papers and manuscripts on Dynamics.

The labour involved has been very great, for less than one-quarter of the contents of this volume had been published before: and much of the new matter is of great interest, especially the series of manuscripts on a method of integrating total and partial differential equations which had been suggested by Hamilton's dynamical researches, and which he called the "Calculus of Principal Relations". From another manuscript paper it appears that the discovery of the difference between phase-velocity and group-velocity, hitherto attributed to Stokes, was made by Hamilton as far back as 1839.

* *The Mathematical Papers of Sir William Rowan Hamilton, Volume II. Dynamics.* Pp. xv, 656. 70s. net. 1940. (Camb. Univ. Press.)

When Hamilton died in 1865, at the age of sixty, his rank in the hierarchy of the great men of science was very uncertain. His name was at that time associated chiefly with the discovery of quaternions: and whether quaternions had much value was a subject of discussion, and often of acrimonious controversy, for the next fifty years, the general trend of opinion becoming on the whole less favourable to them. The nadir of Hamilton's reputation was touched about the beginning of the present century: since when, there has been a steady movement in the reverse direction: one after another, the significance of his great innovations has been appreciated: and to-day he is placed among the greatest, in the class inferior only to Newton.

The history of these fluctuations of opinion is instructive, and reflects closely the development of mathematics and theoretical physics in the last eighty years. Let us take first the nineteenth century.

Quaternions were discovered in 1843, the processes of thought which led to this event being described fully in Graves' *Life* of Hamilton. The original idea was simply to find for three-dimensional space an analogue to the Argand representation of a point of a plane by an ordinary complex quantity, which though involving two real numbers obeys the same algebraical laws as a single number. Similarly, it was conjectured, a point in space of more than two dimensions might be represented by a *polyplet*, or set of numbers behaving like a single one. Graves shows us the younger members of the household at Dunsink sharing affectionately in the hopes and disappointments of their illustrious parent as the investigation proceeded. William Edwin (aged nine) and Archibald Henry (eight) used to ask at breakfast, "Well, Papa, can you *multiply* triplets?" Whereto he was obliged to reply, with a sad shake of the head, "No, I can only *add* and subtract them." But on 16th October, 1843, as he was walking into Dublin from Dunsink along the Royal Canal, he realised that the geometrical operations of three-dimensional spaces required for their description not triplets but quadruplets, and that the multiplication of the quadruplets must be non-commutative. An electric current seemed to close: and the sparks that fell from it were the fundamental equations:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad ijk = -1.$$

The news spread quickly, and led to a wave of interest amongst people of rank and fashion, in Dublin at any rate, something like the boom in Relativity in the early 1920's in London, when Lord Haldane invited Einstein to meet at luncheon the Archbishop of Canterbury. Hamilton was buttonholed in the street by members of the Anglo-Irish aristocracy with the question, "What the deuce *are* the Quaternions?" To satisfy them, he published the delightful *Letter to a Lady*, in which he explained that the term " occurs, for example, in our version of the Bible, where the apostle Peter is described as having been delivered by Herod to the charge of four

quaternions of soldiers. In the *Paradise Lost*, Adam is represented by Milton as uttering the invocation :

> ' Air, and ye Elements, the eldest birth
> Of Nature's womb, that in *quaternion* run
> Perpetual circle manifold. . . .'

And, to take a lighter and more modern instance from the pages of *Guy Mannering*, Scott represents Sir Robert Hazelwood of Hazelwood as loading his long sentences with ' triads and quaternions '."

From this time until his death twenty-two years later, Hamilton's chief interest was to develop this new calculus. They were mostly sad and lonely years, owing to the frequent illnesses and absences of his wife. When I lived at Dunsink in 1906-12, many of the neighbours, who had known Hamilton personally, used to tell strange stories of him. He worked all day in the large dining-room of the Observatory house, into which from time to time his cook passed a mutton chop. After his death, scores of mutton chop bones on plates were found sandwiched among his papers.

The discovery of quaternions would not be regarded to-day as Hamilton's most splendid achievement, but it was undoubtedly of very great importance, for thereby he introduced into pure mathematics the revolutionary idea of a non-commutative algebra,* and developed in detail what is in some respects the simplest algebra of this type,† and moreover showed that it could be a useful tool in theoretical physics. The question we have now to answer is, Why then did the Hamiltonian school, so gloriously started, make no progress to speak of in the next half-century?

One reason is, I think, that those who regarded themselves as Hamilton's chosen disciples—the Edinburgh quaternionists, Tait and Knott—were really physicists, and did not possess that instinct for the generalisation of a theory which is characteristic of the mathematician : they did not lift up their eyes from the particular type of non-commutative algebra found by Hamilton, to non-commutative algebras in general ; but, on the contrary, made it a test of orthodoxy to deviate in no particular from the dispositions of the master. Since he had already worked out the general theory of quaternions, there was little left for them to do except to publish examples of its use in theoretical physics ‡ ; and as all these examples

* Other types of algebra quickly followed : Boole's *Algebra of Logic* (1847) retains the commutative law, but has $x^2 = x$ and has complete duality between addition and multiplication, which are distributive with respect to each other. The Theory of Matrices, which is non-commutative, was discovered by Cayley (1855), but is the same in essence as Hamilton's theory of Linear Vector Operators, which he gave in his *Lectures on Quaternions* in 1852. Non-associative algebras, in which $a \,.\, bc$ is not necessarily equal to $ab \,.\, c$, were suggested by De Morgan. Grassmann's *Ausdehnungslehre*, the first sketch of which appeared the year after the publication of Hamilton's theory, should also be mentioned.

† For the only linear associative algebras over the field of real numbers in which division is uniquely possible are the field of real numbers, the field of ordinary complex numbers, and real quaternions.

‡ I include geometry in theoretical physics, as is right *pace* Whitehead and Russell.

could have been done by ordinary algebra, the performance was not very impressive.* Then those who were in the outer circles of Hamilton's influence—*e.g.* Willard Gibbs in America and Heaviside in England—wasted their energies in devising bastard derivatives of the quaternion calculus—dyadics and vector-analysis—which reproduced its most regrettable feature, namely the limitations imposed by its close association with the geometry of ordinary space, and which represented no advance, but rather a retrogression, from the point of view of general theory. An *International Association for the promotion of the study of Quaternions and allied systems of Mathematics* was founded, and produced a useful bibliography of the subject, but the actual advances in general non-commutative algebra were made by workers outside the direct Hamiltonian tradition.

I remember discussing in 1900 with A. N. Whitehead whether quaternions and other non-commutative algebras had much of a future as regards applications to physics. Whitehead remarked that while all the physics then known could be treated by ordinary algebra, it was possible that new fields in physics might some day be discovered for which non-commutative algebra would be the only natural representation.

In that very year, the advance was made by which this anticipation came ultimately to be fulfilled. Planck, in his famous study of pure temperature radiation, was led to introduce the Quantum h, and so to found the Quantum theory, which has been the chief instrument of progress in physics during the present century. Now h is a quantum of *Action* : and Action was a central conception in Hamilton's system of dynamics : thus, while there was as yet no opening for quaternions, the Hamiltonian ideas on dynamics began to come into prominence.

But very slowly. When my book *Analytical Dynamics* was published in 1904, I was criticised severely for devoting a large part of it to such topics as the duality between coordinates and momenta, contact-transformations, Poisson-brackets, integral-invariants, action, and so forth—mere mathematical playthings, as the critics called them.

The good work, however, went on : and the discovery of (special) relativity in 1905 brought quaternions into a new prominence, for Cayley had shown in 1854 that quaternions can be applied to the representation of rotations in four-dimensional space, and, as Felix Klein remarked, Cayley's result yielded a particularly elegant expression for the coefficients of the most general Lorentz transformation. Moreover, the new discoveries again emphasised the importance of Action, which is relativistically-invariant and a scalar, and therefore fundamental in relativistic physics.

* Though it must be acknowledged that now and then, *e.g.* in Tait's proof of Ampère's laws for the ponderomotive force between elements of electric currents (*Proc. R.S.E.*, 1873-4) and in Heaviside's beautifully simple expression for the Maxwell stresses in an electric field (*Phil. Trans.*, 1892), quaternions made a very good showing.

The meeting of the British Association at Dublin in 1908 helped to revive interest in Hamilton by bringing many English mathematicians and physicists to the scene of his labours : among them Hobson, who read a paper " On Sir W. R. Hamilton's Fluctuating Functions ",* in which Hamilton was revealed in an unexpected light as a pioneer in the Theory of Functions of a Real Variable.

Meanwhile, quantum theory was being developed, and the workers in that field were coming to realise that Hamilton's dynamical conceptions must form the basis of all rules of quantisation. The fundamental equations of the " old " or pre-1925 quantum theory were that certain integrals of the type $\int p\,dq$, where q and p are Hamiltonian conjugate variables (a coordinate and the corresponding momentum) must be exact multiples of Planck's constant.

The other side of Hamilton's work—the non-commutative algebra—was brought into quantum theory in 1925 by Heisenberg, Born and Jordan, who showed that the ordinary Hamiltonian equations of classical dynamics :

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q},$$

are still valid in quantum theory, provided the symbols q, p, H are now interpreted as *operators* whose products do not commute : they do in fact satisfy the equation

$$pq-qp=\frac{h}{2\pi i}.$$

Thus the foundation on which modern quantum-theory is built may be described as a combination of two elements both originating with Hamilton—the dynamical theory and the non-commutative theory. The most recent work, such as the theories of the neutrino and the meson, has been developed entirely by non-commutative methods.

Time has amply vindicated Hamilton's intuition of the duality between generalised coordinates and generalised momenta. This was strikingly shown in 1927 when Heisenberg discovered the *Principle of Uncertainty*, which is usually stated thus : the more accurately the coordinate of a particle is determined, the less accurately can its momentum be known, and *vice versa*, the product

* Afterwards published in *Proc. L.M.S.* (2) 7 (1909), 338.

Mention may here be made of an incident in the history of geometrical optics. Bruns—the author of the famous theorem on the non-existence of further algebraic integrals in the problem of three bodies—published in 1895 a method of describing and calculating the aberrations of optical instruments, depending on a function which he called the *eikonal*. This, as Klein pointed out, was essentially nothing more or less than the " characteristic function " of Hamilton's optical papers : and since then the Hamiltonian method has been used extensively in practical calculations relating to the aberrations of the symmetrical optical instrument. From a very different point of view, the optical papers came to be recognised by geometers as the origin of the theory of the general rectilinear congruence.

of the two uncertainties being of the order of Planck's constant h.* Within the last two or three years, Professor Max Born has applied the principle of duality in an entirely new direction by his *Principle of Reciprocity*, which offers a prospect of unifying Quantum Theory with Relativity. The fundamental idea is that relativity effects are most in evidence when we consider the largest things—the expanding universe, etc.—while quantum effects are most in evidence in the smallest things: Born regards these as two extreme types of physical system, in one of which the space coordinates may take very large values, while in the other—nuclear phenomena—the spatial regions are very small but the energy and momenta concerned may be large. Thus between cosmical phenomena and nuclear phenomena there exists a duality corresponding precisely to Hamilton's duality between coordinates and momenta. In the hands of Born and his group of research students in Edinburgh, this idea is promising to yield the secret of the mysterious number 137.

Quantum-mechanical workers have generally tended to regard matrices rather than quaternions as the type of non-commutative algebra best suited to their problems: but the original Hamiltonian formulae keep cropping up. Thus the "spin matrices" introduced by Pauli in 1927, on which the quantum-mechanical theory of rotations and angular momenta depend, are simply Hamilton's three quaternion units i, j, k. Professor Conway has shown in an interesting paper † that quaternion methods may be used with advantage in the discussion of Dirac's equation for the spinning electron: and the formalism of 1843 may even yet prove to be the most natural expression of the new physics. E. T. W.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—In connexion with the article on Hamilton which you print elsewhere in this number of the *Gazette*, it would be of interest to ascertain whether Hamilton has any living descendants. He had only one grandchild, his daughter's son, John Rowan Hamilton O'Regan, born in 1870, who, when Graves' *Life of Hamilton* was published in 1889, was a scholar of Clifton College. Whether Mr. O'Regan is still living, and whether he married and had issue, I do not know. Perhaps through some Old Cliftonian, or organisation of Old Cliftonians, it might be possible to trace him and so settle the question.—I am, SIR, Yours truly,

E. T. WHITTAKER.

* This discovery led to the publication of an enormous amount of nonsense about determinism and free-will. Actually the uncertainty-principle has nothing whatever to do with determinism or free-will: it simply expresses the fact that the classical concept of a particle (*i.e.* a very small body having a definite position and velocity) does not represent anything in nature: what is realised in nature is the quantum concept of a particle, which is something quite different.

† *Proc. R.S.*, 162 (1937), 145.

THE EVOLUTION OF COSMOLOGY.

By G. J. Whitrow.

Thales of Miletus (640-546 B.C.) was the first cosmologist. Despite the naïvety of his particular theory, that "everything is water", he was the first to formulate the hypothesis that, in all the multiplicity and diversity of its aspects, the World is One. Although the Babylonians are believed to have conceived the idea of an "Immutable Necessity" dominating all things, as far as we know they had no clear idea of the unity of the universe.

After Thales, other philosophers replaced his fundamental "water" by "air" and "fire" and, in particular, by the much more sophisticated concept of the "infinite". By this was understood the notion of cosmic matter diffused throughout (and indeed identical with) space. Pythagoras (*circa* 530 B.C.) conceived the idea of the condensation of this primeval infinite around "monads" as centres so that "points" were delimited. This idea was, doubtless, suggested to him by his astronomical observations, for the stars appear to be an assemblage of "points". Pythagoras' doctrine, that "things are numbers", can be regarded as the first of all atomic theories. By a number he seems to have meant a set of "points" in definite order and formation. He did not regard a "point" as dimensionless. It was "unity having position". His theory can be expressed more lucidly in the following words, "every physical object is composed of spatial units (points), which taken together constitute a number". Pythagoras' theory was a tremendous impetus to mathematical research for it was not only a doctrine but also a "programme". Previously mathematics had consisted merely of disconnected topics in arithmetic and empirical geometry together with a few isolated theorems. By uniting geometry and arithmetic Pythagoras founded Mathematics as we now understand it, *i.e.* as a self-consistent unified discipline, and thus through him we can trace the history of this science back to its origins in Cosmology.

Pythagoras' views were developed in great detail by subsequent Greek thinkers and resulted in the famous doctrine of "place", which dominated European thought from the time of Aristotle to that of Galileo. According to this theory every object in the universe has a natural place, which it either occupies or else seeks. Motion is therefore "absolute". The progress of physics since the scientific renaissance in the sixteenth century has been largely due to the gradual overthrow of this idea.

The first modern thinker to consider seriously the possibility of rejecting this idea was the famous Polish astronomer and mathematician, Copernicus. In his great work, *De Revolutionibus Orbium Caelestium*, published in 1543, Copernicus adopted the view that, whereas the sun appears to go round the earth, it is simpler *mathematically* to assume that the earth goes round the sun. Till then astronomy had been based on a refinement of Aristotle's own World-model, known as the Ptolemaic system, according to which the

earth was central and immovable. It is important to remember that, with the instruments of the day, there were no known astronomical phenomena which could not be fitted into this system. To the many serious objections which were advanced against his theory Copernicus could only plead mathematical simplicity. In the end this attitude prevailed, but the reconciliation with the observed facts of the idea of a moving earth, *which seemed to contradict all one's physical intuitions*, implied the acceptance of the revolutionary doctrine of the relativity of motion.

The old ideas and the new were most fruitfully combined in Newton's *Principia* (1st edition, 1687). Newton recognised the relativity of uniform motion in a straight line and based his natural philosophy on the assumption that the laws of nature appear to be the same from all standpoints in uniform motion. Nevertheless, in consequence of a celebrated experiment with a rotating bucket of water, Newton concluded that rotational motion is absolute and thus he was obliged to retain the concepts of an absolute space and an absolute time. It was his great contemporary and adversary, Leibniz, who first enunciated clearly the modern point of view. For him " space and time, extension and motion are not things but methods of contemplating them ". Space he defined as " the order of co-existences ", time as " the order of succession of phenomena ".

Of far-reaching significance for modern Cosmology was the discovery by the Danish astronomer, Römer (1675), of the non-instantaneous transmission of light. This first revealed the importance in natural philosophy of the observer, since it follows that, to use two useful terms coined by Professor E. A. Milne, the World-picture (*i.e.* the World-as-seen) at a definite epoch is not identical with the World-map (*i.e.* the World-as-it-" is ") at that epoch. From the time of Copernicus there has been a gradual discarding of all non-relativistic concepts in Cosmology. Nevertheless, until the beginning of the present century physicists felt obliged to admit, in the same breath, both the notion of the relativity of uniform motion and the existence of an absolute frame of space and time which, after Maxwell enunciated his famous electromagnetic theory of light, took the form of the luminiferous " ether ". Looking back from our present standpoint, the year 1887 appears as one of the crucial years in the evolution of Cosmology. In that year Michelson and Morley performed their celebrated experiment with light rays to determine the absolute motion of the earth through the ether of space. To the surprise of the scientific world, each time that Michelson and Morley performed their experiment they obtained a null result.

In 1905, following fruitful preliminary spade-work by Larmor and Lorentz, Einstein in a famous memoir, " On the Electrodynamics of Moving Bodies ", explained this null result by rejecting the existence of an ether and of an absolute frame of space and time. Instead, the notion of the relativity of uniform motion was extended to electromagnetic as well as ordinary dynamical and gravitational phenomena. In this way not only was he able to explain why

Michelson and Morley obtained a null result but also in the case of certain other results in physics he could give simpler and more convincing explanations than the current ones. Nevertheless, on the nature of rotational motion and of accelerated motion in general Einstein's *Special Theory of Relativity* threw no light. Consequently in the following decade he developed a *General Theory of Relativity* (published in 1916) in which he postulated that the laws of nature must be the same referred to all coordinate systems in all types of relative motion. He combined space and time to form a four-dimensional manifold, space-time, and thereby developed a geometrical method for solving all problems of a gravitational (*i.e.* non-electromagnetic) character.

In his 1916 memoir Einstein confined the scope of his investigations to so-called "local" gravitational situations. But, in the following year, he applied his new methods to the problem of world gravitation, *i.e.* to the problem of the large-scale distribution of matter in the universe, as determined by its own gravitational field. In seeking a solution he was, of course, guided by the general features which astronomers were inclined to assign to the then observable region of the universe.

The pioneer of modern observational Cosmology was Sir William Herschel (1738-1822), well known to most people as the discoverer of the planet Uranus. Previous astronomers had confined their attention almost entirely to the solar system. Observation of the stellar system lagged behind speculation. Giordano Bruno (burnt by the Inquisition in 1600) likened the stars to our own sun, but it was not till the middle of the eighteenth century that views were expressed about the nature of the Milky Way or Galaxy. Then several thinkers, including the philosopher Kant, put forward the theory that the Galaxy is a disc-shaped distribution of stars. Herschel, however, was the first to seek definite observational evidence. Not only did his observations over many years strengthen this theory but he was also led to the view that there exist many other independent stellar systems of similar type, which he called "nebulae". Two of these, one of which is the Great Nebula in Andromeda, are just visible to the naked eye.

With the rapid advance in observational astronomy in the nineteenth century it became clear that there are different types of nebulae and that not all are necessarily outside the Galaxy. However, some, mainly of spiral shape, were believed to be resolvable into assemblages of stars and, therefore, were indeed likely to be independent systems. The most convincing proof of this, however, first came from measurements of the radial motion of these objects.

In 1842 an Austrian physicist, Doppler, expressed the view that, if light from any source, *e.g.* a star or a nebula, be split up by a prism into a spectrum, then the entire spectrum is shifted towards the violet if the source is approaching and towards the red if it is receding. In either case from the amount of shift the radial velocity of the source can be calculated by a simple formula.

In 1914 Dr. V. M. Slipher, as a result of his pioneer work on the radial velocities of spiral nebulae, announced that of fourteen spirals which he had studied, all but two were receding with large velocities. It thus became clear that, in all probability, the spirals are independent systems rapidly receding from our neighbourhood.

Despite this discovery, however, when, in 1917, Einstein first applied General Relativity to the problem of world-structure it still seemed that the average velocity of matter in the universe was small compared with that of light so that, to a first approximation, the general distribution of matter can be regarded as relatively static. Moreover, Einstein decided that a model of the universe in which matter was distributed homogeneously would give some insight into the general nature of the world, neglecting local irregularities and condensations (stars, etc.). For various reasons Einstein concluded that such a smoothed-out universe could not be infinite but must be " finite yet unbounded ".

Einstein constructed his model in accordance with certain gravitational equations. Shortly afterwards the Dutch astronomer, de Sitter, pointed out that there existed another static model satisfying these equations, but that this model, strictly speaking, contained no matter. If, however, matter were introduced into it then it would no longer remain static but would begin to expand.

Since 1917 enormous progress has been made in our observational knowledge of spirals or extra-galactic nebulae, as they came to be called. In 1925 Dr. E. Hubble succeeded in partially resolving two of the brightest into stellar systems, and in 1929 he discovered, with the aid of the 100-inch telescope at Mount Wilson Observatory, that the more distant a nebula appears to be the more its spectrum is shifted towards the red. If this shift is due to recession then, according to Hubble, the furthest nebulae are receding fastest according to a law of simple proportionality between distance and velocity. More recent investigations have led to the discovery of red-shifts indicating enormous velocities of recession up to one-tenth that of the speed of light (186,000 miles per second). Moreover, the further we peer into space the more extra-galactic nebulae we observe. So far there has been no sign of any thinning out. Consequently, although it still appears that, to a first approximation, the universe is homogeneous and isotropic, there is no longer the same disposition to assume, without further discussion, that it is static.

It has, of course, been clear from the first that the actual universe must be, in some sense, intermediate between the Einstein and de Sitter idealised systems. It might possibly be obtained by pumping matter into the originally empty de Sitter model or else by allowing condensations to form in the originally homogeneous Einstein model. In the former case, as we have remarked, it was already known in 1917 that the model would expand, but it is a curious historical fact that not until 1930 did anyone investigate the consequences of allowing condensations to form in the Einstein model. In that year it was discovered by Sir Arthur Eddington that this model too is

unstable and that, if submitted to a slight disturbance, it would either start to expand or else to contract.

In 1922 a Russian, Friedmann, first discovered non-static solutions of Einstein's gravitational equations for a world-model, but his work was ignored at the time and it was not until these solutions were rediscovered by the Belgian mathematician, Lemaître, in 1927 that they attracted general attention. After Hubble announced his law of red-shifts and Eddington discovered the instability of the Einstein model there was a rapid revival of interest in the theory of expanding models of the universe.

A new twist was given to the whole subject, however, in a series of fundamental investigations initiated by E. A. Milne in 1932.

In the General Theory of Relativity the " shape " of space-time is determined by the amount and distribution of matter present. This idea of space-time possessing a structure seems suspiciously like a readmission by the back door of the old discredited ether. Moreover, those experimental tests which appear to justify the application of this theory to problems of local gravitation (*e.g.* inside the solar system) do not necessarily justify its application to problems of world gravitation. Indeed as Milne has remarked, " When General Relativity comes to discuss the whole universe, it calculates the pull of the whole rest of the universe on the same empirical principles as it used in local gravitational situations. In so doing it counts the rest of the universe twice over, once in formulating the laws of motion and once again in calculating the ' pull '."

Milne, therefore, proposed a fresh start. He began by pointing out that, in general, *any* distribution of particles, moving with uniform velocities in a finite region of space, would eventually be found to be receding from one another according to Hubble's relation, the fastest having got furthest. This simple kinematical model was criticised, because it neglected the effect of gravity, but Milne then pointed out that it was intended merely as a hint towards the formulation and application of an entirely new *method*, which has since been called Kinematical Relativity.

This discipline is based on the observer's awareness, in principle, of an ordered, continuous set of events at himself. A correlation can be set up between these events and the continuum of real numbers. Such a correlation is called a clock " arbitrarily graduated ". Two observers are said to possess congruent clocks, and thus to be " equivalent ", if by simple experiments with light-signals they can calibrate their clocks into agreement. A continuous system of equivalent observers in line is called a " linear equivalence " and a three-dimensional isotropic pencil of linear equivalences made up of observers attached to fundamental " particles " and distributed so that each observer sees himself central in the same world-picture, is said to constitute a " substratum ". A substratum can be compared with the universe of observation (smoothed-out), in which we have, as yet, no reason for supposing that our own or any other nebula occupies a privileged position.

It has been proved that each fundamental equivalent observer in a substratum can graduate his clock so as to describe all the others as moving away from himself with uniform radial velocity, distance being measured in terms of clock-readings by a *convention* concerning light-signals. Alternatively, he can graduate his clock so as to describe the whole system as static. If we assume that in the actual universe the nebulae are the bricks out of which the world is built up, we can identify them, in the model substratum, with the fundamental " particles " to which equivalent observers are attached. This identification throws an entirely new light on the questions, " Is the universe ' really ' static, or is it ' really ' expanding, and if so what is its ' age '? " For we now see, in a way which was not hitherto quite clear, that our answers presuppose a definite *choice* of a system of measurement. If, however, we assume, in virtue of the existence of red-shifts in the spectra of extra-galactic nebulae, that the uniform time of optical and atomic phenomena is that according to which the universe is expanding uniformly; then, although the number of nebulae in the model is infinite, they are all contained in a shell of finite radius, which increases with the epoch. Moreover, by means of Hubble's law, we can immediately assign to the world an " age " of the order of two thousand million years.

Now, if this result were not supported by any independent arguments, it would be extremely interesting but not necessarily more important than other answers obtained by adopting alternative systems of measurement. It is, however, of the utmost significance because entirely independent methods of calculating upper limits to the " age " of rocks and meteorites, by means of radioactive phenomena, again lead us to assign a figure of the order two thousand million years to the " age " of the universe.

In answer to those critics who maintained that he had neglected gravitation, Milne has shown, by investigating the *dynamical* motion of a free particle in the substratum, that the fundamental particles themselves are free, *i.e.* not constrained, so that their kinematical motion is that prescribed by their own gravitational field. Moreover, the uniform time of dynamics is found to differ from that of optics and atomic physics. It is, in point of fact, that graduation of the fundamental observers' clocks according to which the substratum is described as static.

The programme of research opened up by this pioneer work is vast and embraces the possibility not only of a new and closer analysis of problems of world-gravitation but also of those concerned with local gravitation, of the condensation of matter into stellar systems and also of problems of an electromagnetic and atomic character, which have so far resisted the efforts of Einstein and his followers. Much has already been done, mainly by Milne himself, but the field of application of his new methods is still far from exhausted. Thus Cosmology has taken on a new lease of life and appears to be both the oldest and the youngest of the sciences. G. J. W.

POINTS CONNECTED WITH THE COMPLETE QUADRILATERAL.

By N. M. Gibbins.

1. The stimulus that provoked this article was the following proof of the fact that the middle points of the diagonals of a complete quadrilateral are collinear.

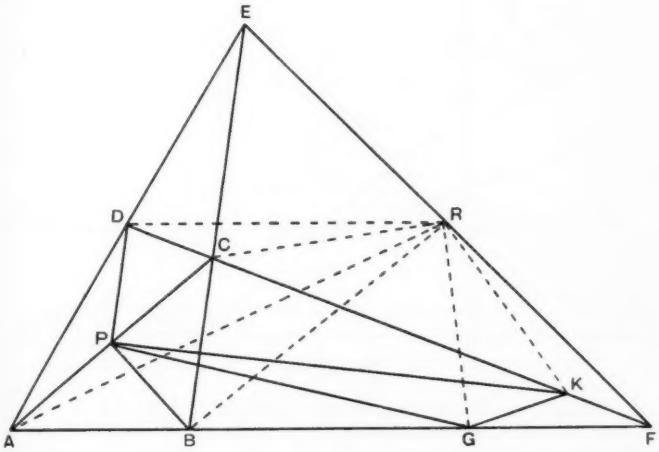


Fig. 1.

Let $ABCDEF$ be the complete quadrilateral. In FB cut off $FG = BA$ and in FD cut off $FK = CD$. Let P be the midpoint of AC and R the midpoint of EF. Complete the figure.

Then $$PGF = PAB = \tfrac{1}{2}CAB$$

and $$PKF = PDC = \tfrac{1}{2}ADC.$$

Hence, by addition, $PGFK = \tfrac{1}{2}ABCD$;

that is, $$PGK + FGK = \tfrac{1}{2}ABCD.$$

Similarly, if Q be the midpoint of BD,

$$QGK + FGK = \tfrac{1}{2}ABCD.$$

Also $$RGF = RAB = \tfrac{1}{2}EAB,$$

$$RKF = RDC = \tfrac{1}{2}EDC.$$

Hence, by subtraction,

$$RGK + FGK = \tfrac{1}{2}ABCD.$$

Hence P, Q, R lie on a line parallel to GK

2. The foregoing proposition could equally well be proved by cutting off $EG' = DA$ and $EK' = CB$, whence PQR is parallel to $G'K'$. Hence $G'K'$ is parallel to GK.

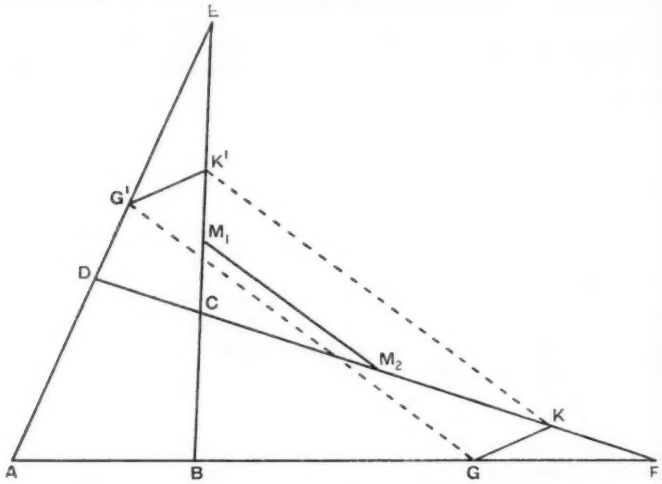


FIG. 2.

Applying this idea to the convex quadrilateral $BDEF$ of which BE and DF are the internal diagonals, and A one external diagonal extremity, we have $AG = BF$ and $AG' = DE$. Hence $G'G$ is parallel to M_1M_2, where M_1 and M_2 are the midpoints of BE and DF, that is, of CK' and CK. Hence $G'G$ is parallel to $K'K$, so that $GKK'G'$ is a parallelogram.

We may call G the internal isotomic conjugate of B with respect to AF, and similarly for K, K', G'.

3. In BA produced cut off $AL' = BF$; then $BL' = AF$. G' and K' are as in Fig. 2, so that $BK' = CE$. We have (Fig. 3)

$$PBL' = PAF = \tfrac{1}{2}CAF,$$
$$PBK' = PCE = \tfrac{1}{2}ACE.$$

Hence $$PBL' + PBK' = \tfrac{1}{2}EAFCE,$$

that is, $$BK'L' - PK'L' = \tfrac{1}{2}EAFCE.$$

Similarly, if Q is the midpoint of BD,

$$BK'L' - QK'L' = \tfrac{1}{2}EAFCE.$$

Hence $K'L'$ is parallel to PQ. But $K'G'$ is parallel to PQ.

Hence $K'G'L'$ is a straight line. L' may be called the external isotomic conjugate near to A of B with respect to AF. The external isotomic conjugate near to F is got by producing BF to X, say, so that $FX = AB$.

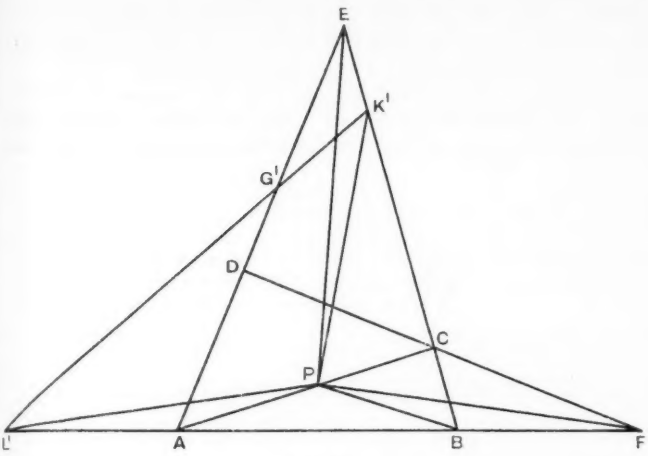


FIG. 3.

4. Similarly, if DA be produced to L so that $AL = DE$, KGL is a

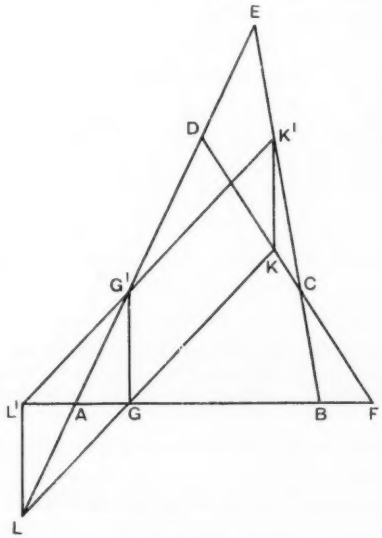


FIG. 4.

straight line.

Since $AL' = BF = AG$

and $AL = DE = AG'$,

$GG'L'L$ is a parallelogram. Hence also so is $KLL'K'$.

COROLLARY. Hence the join of A to the centre of $GKK'G'$ bisects GG'. If then $BF = DE$, $AG = AG'$ and $GKK'G'$ is a rectangle.

5. If now we mark off the three isotomic conjugates on each side

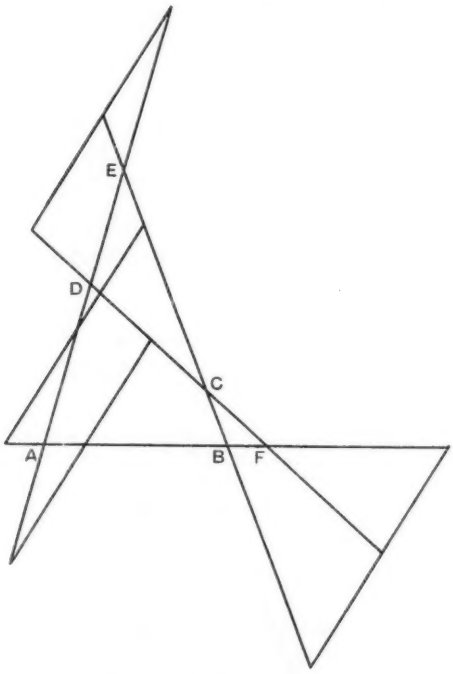


FIG. 5.

of the quadrilateral, making twelve new points in all, they lie three by three on four lines which are all parallel to the join of the mid-points of the diagonals; and we can construct from them eighteen parallelograms. N. M. G.

GLEANINGS FAR AND NEAR.

1313. . . . There is no branch of study—no, not mathematics itself—which forms a finer mental discipline than Ethics.—Canon Peter Green, *The Problem of Right Conduct*, p. 10. [Per Mr. A. F. Mackenzie.]

ON THE DIAGONAL TRIANGLES OF CERTAIN QUADRILATERALS ASSOCIATED WITH A TRIANGLE.

By F. H. V. Gulasekharam.

Mr. Gibbins' paper on the "Feuerbach Quadrilateral" (*Math. Gazette*, No. 249) deals with a delightful topic. The device employed is certainly ingenious. May I, however, be permitted to discuss the topic from an entirely different point of view, but with the same object in mind, namely of stimulating our pupils' interest in Analytical Geometry. I shall also bring out some interesting properties of the diagonal triangles and the diagonal-point triangles of certain quadrilaterals associated with a triangle, employing *general* rather than *special* methods to arrive at the results.

1. *The Feuerbach points* π, π_1, π_2, π_3 *and the corresponding tangents* t, t_1, t_2, t_3.

The following well-known equation of a circle is of great importance :

If ρ_1^2, ρ_2^2, ρ_3^2 be the powers of A, B, C with respect to a circle, then the *areal* equation of the circle (with ABC as the triangle of reference) is

$$(\rho_1^2 x+\rho_2^2 y+\rho_3^2 z)(x+y+z)-\Sigma a^2 yz=0. \quad \ldots\ldots\ldots\ldots(1.1)$$

What are the values of ρ_1^2, ρ_2^2, ρ_3^2 for (i) the in-circle, (ii) the nine-points circle of $\triangle ABC$? The answer leads us at once to the equation of the radical axis of the two circles, in the form

$$\Sigma(a-b)(a-c)x=0,$$

or

$$\Sigma x/(b-c)=0. \quad \ldots\ldots\ldots\ldots(1.2)$$

Now an advanced pupil knows that the envelope equation of any in-conic of ABC is $fmn+gnl+hlm=0$, with its centre at $(g+h, h+f, f+g)$, and that for the in-circle

$$f:g:h=s-a:s-b:s-c.$$

Writing the envelope equation of the in-circle in the form

$$\Sigma(s-a)/l=0, \quad \ldots\ldots\ldots\ldots(1.3)$$

we see that the radical axis given by (1·2) touches the in-circle. Hence the in-circle and the nine-points circle touch each other. The point of contact is the *Feuerbach point* π. The equation of the common tangent t at π is given by (1·2). It is a simple exercise for the pupils to show that the coordinates of π are

$$[(s-a)(b-c)^2,\ (s-b)(c-a)^2,\ (s-c)(a-b)^2]. \quad \ldots\ldots\ldots(1{\cdot}4)$$

By changing a into $(-a)$ in the foregoing, we prove that the nine-points circle touches the ex-circle (centre I_1) at the Feuerbach point π_1 whose coordinates, and the equation of the common tangent t_1 at π_1, are obtained by replacing a by $(-a)$ in (1·4) and (1·2).

Similarly for (π_2, t_2) and (π_3, t_3).

It will be found that the coordinates of π, π_1, π_2, π_3 and the equations of the corresponding tangents take *simpler forms when we take the medial triangle $A'B'C'$ of ABC for the triangle of reference.*

For, the powers of A', B', C' with respect to the in-circle of ABC are

$$\tfrac{1}{4}(b-c)^2,\ \tfrac{1}{4}(c-a)^2,\ \tfrac{1}{4}(a-b)^2$$

respectively.

Hence using (1·1), the radical axis of the in-circle of ABC and the nine-points circle of ABC (the latter being the circumcircle of $A'B'C'$), is

$$(b-c)^2x+(c-a)^2y+(a-b)^2z=0. \quad \text{.....(1·5)}$$

Now the point-equation of any circum-conic of the triangle of reference is $f/x+g/y+h/z=0$, and the irrational form of its tangential equation is

$$\surd(fl)+\surd(gm)+\surd(hn)=0,$$

and the point of contact of a tangent $lx+my+nz=0$ is

$$[\surd(f/l),\ \surd(g/m),\ \surd(h/n)]. \quad \text{.....(1·6)}$$

For the circumcircle of $A'B'C'$,

$$f:g:h=a^2:b^2:c^2.$$

Hence we see that the radical axis (1·5) touches the circle $A'B'C'$, and that therefore the nine-points circle and the in-circle of $\triangle ABC$ touch each other at

$$\pi\equiv[a/(b-c),\ b/(c-a),\ c/(a-b)]; \quad \text{.....(1·7)}$$

the corresponding common tangent t at π is given by equation (1·5).

Changing a into $(-a)$ in the foregoing, as before, we get

$$\pi_1\equiv[-a/(b-c),\ b/(c+a),\ -c/(a+b)], \quad \text{.....(1·8)}$$

while t_1 is given by

$$(b-c)^2x+(c+a)^2y+(a+b)^2z=0. \quad \text{.....(1·9)}$$

Similarly for (π_2, t_2) and (π_3, t_3).

[To illustrate the advantage of taking $A'B'C'$ for the triangle of reference, the following may be set as examples for the pupils to tackle :

(i) t_2, t_3 meet on the internal bisector of the angle $B'A'C'$, while t, t_1 meet on the external bisector ; similarly for the other vertices of the Feuerbach quadrilateral $(tt_1t_2t_3)$.

(ii) The triangle $A'B'C'$ is in perspective with the triangle formed by the lines $A'\pi_1$, $B'\pi_2$, $C'\pi_3$.

(iii) The vertices of the *diagonal-point triangle* of the quadrilateral $\pi\pi_1\pi_2\pi_3$ are the points $(0, a^2-b^2, c^2-a^2)$, etc., and the triangle is self-polar with respect to the nine-points circle of $\triangle ABC$.

(iv) The *diagonal-point triangle* of $\pi\pi_1\pi_2\pi_3$ coincides with the *diagonal* triangle of the quadrilateral $(tt_1t_2t_3)$, and is in perspective with $\triangle A'B'C'$, the pole of perspective being a point on the nine-

points circle of $\triangle ABC$, and the axis of perspective is the line joining the centroid and the *symmedian point of* $\triangle A'B'C'$.]

2. *The in-conic of* $\triangle ABC$ *touching the sides of the Feuerbach quadrilateral* $(tt_1t_2t_3)$.

The envelope equation of any in-conic which touches the sides of $\triangle ABC$ is

$$f/l+g/m+h/n=0.$$

Is it possible to find $f:g:h$ such that the conic may also touch each of the lines t, t_1, t_2, t_3? There should be no difficulty in discovering that $f=g=h$ is the only solution.

Hence the Feuerbach quadrilateral is circumscribed to the in-conic

$$1/l+1/m+1/n=0,$$

or
$$mn+nl+lm=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}1)$$

The irrational form of the corresponding point-equation is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}2)$$

The point of contact of any tangent $lx+my+nz=0$ is

$$(m+n,\ n+l,\ l+m).$$

We can now proceed to study this conic and show that it touches BC, CA, AB at A', B', C' respectively, and has its centre at the centroid G of $\triangle ABC$. This conic is the *Steiner ellipse* of $\triangle ABC$ and is in itself an interesting topic for study.

The lines t, t_1, t_2, t_3 are the four common tangents of the nine-points circle and the Steiner ellipse of $\triangle ABC$.

The points of contact of the Steiner ellipse with t, t_1, t_2, t_3 are the points ξ, ξ_1, ξ_2, ξ_3 respectively, where

$$\xi\equiv[(b-c)^2,\ (c-a)^2,\ (a-b)^2], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}3)$$

$$\xi_1\equiv[(b-c)^2,\ (c+a)^2,\ (a+b)^2], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}4)$$

and so on, with $\triangle ABC$ as triangle of reference.

With $A'B'C'$ as triangle of reference,

$$\xi\equiv[1/(b-c),\ 1/(c-a),\ 1/(a-b)], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}5)$$

$$\xi_1\equiv[1/(b-c),\ 1/(c+a),\ -1/(a+b)], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2{\cdot}6)$$

and so on.

3. *The diagonal triangle* $X_0Y_0Z_0$ *of the complete quadrilateral* $(tt_1t_2t_3)$.

As pointed out by Mr. Gibbins, the equations of the sides of the triangle are

$$y/\mu+z/\nu=0,\ \ z/\nu+x/\lambda=0,\ \ x/\lambda+y/\mu=0, \quad \ldots\ldots\ldots\ldots\ldots(3{\cdot}1)$$

where
$$\lambda=b^2-c^2,\ \ \mu=c^2-a^2,\ \ \nu=a^2-b^2. \quad \ldots\ldots\ldots\ldots\ldots(3{\cdot}2)$$

Now *for all values* of λ, μ, ν, the vertices of the triangle whose sides are given by (3·1) are the points

$$(-\lambda,\ \mu,\ \nu),\ \ (\lambda,\ -\mu,\ \nu),\ \ (\lambda,\ \mu,\ -\nu). \quad \ldots\ldots\ldots\ldots\ldots(3{\cdot}3)$$

Hence for *all* λ, μ, ν, the triangle is in perspective with $\triangle ABC$, the

pole of perspective being the point (λ, μ, ν) and the axis of perspective the line

$$x/\lambda + y/\mu + z/\nu = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3{\cdot}4)$$

Mr. Gibbins has already drawn attention to certain relations which exist between any two such triangles for which $\lambda + \mu + \nu = 0$. *An additional property is that, when* $\lambda + \mu + \nu = 0$, *the axis of perspective* (3·4) *envelops the Steiner ellipse of* $\triangle ABC$.

In the particular case of $\triangle X_0Y_0Z_0$, *i.e.* when λ, μ, ν are given by (3·2), the axis of perspective (3·4) is also a tangent to the in-conic (of $\triangle ABC$) whose envelope equation is

$$\Sigma(b^2 + c^2 - a^2)mn = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3{\cdot}5)$$

This conic touches BC, CA, AB at the vertices P, Q, R of the pedal triangle of the orthocentre H of $\triangle ABC$, and has its centre at (a^2, b^2, c^2), that is, at the *symmedian point* K of $\triangle ABC$. This is obviously an ellipse. Let us call it the *pedal ellipse* of $\triangle ABC$.

Let us denote by t_0 the axis of perspective $\Sigma x/(b^2 - c^2) = 0$ of the triangles ABC, $X_0Y_0Z_0$.

The Steiner ellipse touches t_0 at the point

$$[1/(c^2 - a^2) + 1/(a^2 - b^2), \text{ etc. } \ldots],$$

i.e. at $[(b^2 - c^2)^2, (c^2 - a^2)^2, (a^2 - b^2)^2]$, which with $\triangle A'B'C'$ as triangle of reference is the point $[1/(b^2 - c^2),\ 1/(c^2 - a^2),\ 1/(a^2 - b^2)]$. Let us call it w' ; then w' *is the fourth point of intersection of the nine-points circle and the Steiner ellipse of* $\triangle ABC$. It is easy to verify that w' is the pole of perspective of the triangles $A'B'C'$ and $X_0Y_0Z_0$.

Again, the pedal ellipse of $\triangle ABC$ touches t_0 at the point

$$[b^2/(c^2 - a^2) + c^2/(a^2 - b^2), \text{ etc.}]$$

with respect to $\triangle ABC$, which with respect to $\triangle A'B'C'$ becomes the point

$$w_0 \equiv [a^2/(b^2 - c^2),\ b^2/(c^2 - a^2),\ c^2/(a^2 - b^2)].$$

Hence w_0 *is the fourth point of intersection of the nine-points circle and the pedal ellipse of* $\triangle ABC$, *the other three points being* P, Q, R.

4. *The diagonal triangles* $X_0Y_0Z_0$, XYZ, $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$ *of the quadrilaterals* $(abct_0)$, $(abct)$, $(abct_1)$, $(abct_2)$, $(abct_3)$ *respectively.*

The quadrilateral $(abct_0)$ is formed by the four common tangents BC, CA, AB and t_0 of the Steiner ellipse and the pedal ellipse of $\triangle ABC$.

The quadrilateral $(abct)$ is formed by the common tangents of the Steiner ellipse and the in-circle of $\triangle ABC$.

The quadrilaterals $(abct_1)$, $(abct_2)$, $(abct_3)$ refer similarly to the Steiner ellipse and each of the ex-circles.

The vertices of the five diagonal triangles are given by (3·3) where λ, μ, ν are given by the following table :

	λ	μ	ν
$X_0\ Y_0\ Z_0$	b^2-c^2	c^2-a^2	a^2-b^2
$X\ \ Y\ \ Z$	$b-c$	$c-a$	$a-b$
$X_1\ Y_1\ Z_1$	$b-c$	$c+a$	$-a-b$
$X_2\ Y_2\ Z_2$	$-b-c$	$c-a$	$a+b$
$X_3\ Y_3\ Z_3$	$b+c$	$-c-a$	$a-b$

...(4·1)

It will be observed that the quadrilaterals $(tt_1t_2t_3)$ and $(abct_0)$ have the same diagonal triangle, viz. $\triangle X_0Y_0Z_0$.

The following properties of the diagonal triangles will now be noted :

(i) In each case, the points A, B, C lie one on each side of the diagonal triangle.

(ii) Each diagonal triangle is in perspective with $\triangle ABC$: the pole of perspective (λ, μ, ν) being a " point at infinity ", the joins of corresponding vertices are parallel.

(iii) Since $\lambda+\mu+\nu=0$, the " x " vertices of *all* the triangles lie on $-x+y+z=0$, that is, on the line $B'C'$. Similarly the " y " vertices lie on $C'A'$, and the " z " vertices on $A'B'$.

(iv) *For the triangle* $X_0Y_0Z_0$,

$$\Sigma(b^2+c^2-a^2)\lambda=0,$$

or

$$\lambda \cot A+\mu \cot B+\nu \cot C=0. \quad \text{.....(4·2)}$$

Hence, the pole of perspective (λ, μ, ν) of the triangles $X_0Y_0Z_0$ and ABC is the point at infinity on the straight line $\Sigma x \cot A=0$, which is the axis of perspective of the triangles PQR and ABC. With the aid of (1·1), this straight line can be shown to be the radical axis of the coaxal system of circles of which the circumcircle, the nine-points circle and the polar circle of $\triangle ABC$ are members. Hence AX_0, BY_0, CZ_0 are at right angles to the *Euler line* of $\triangle ABC$.

Again from (4·2), it is obvious that X_0 lies on the line

$$-x \cot A+y \cot B+z \cot C=0,$$

that is, on the line QR. *Thus we find that* $B'C'$ *and* QR *meet at* X_0 ; $C'A'$ *and* RP *at* Y_0 ; $A'B'$ *and* PQ *at* Z_0.

(v) *For the triangle* XYZ,

$$\Sigma(s-a)\lambda=0. \quad \text{.....(4·3)}$$

Hence the pole of perspective of the triangles ABC, XYZ is the point at infinity on the straight line $\Sigma(s-a)x=0$.

Now if D, E, F be the points of contact of the in-circle with BC, CA, AB respectively, then it is well known that the triangles ABC, DEF are in perspective, the pole of perspective being the *Gergonne point* $\Gamma[1/(s-a), 1/(s-b), 1/(s-c)]$, and the axis of perspective the straight line $\Sigma(s-a)x=0$. Hence, AX, BY, CZ are parallel to this axis of perspective.

Again, from (4·3), we see that the point X lies on the straight line $-(s-a)x+(s-b)y+(s-c)z=0$, that is, on the line EF. Thus we find that $B'C'$ and EF intersect at X; $C'A'$ and FD at Y; $A'B'$ and DE at Z.

(vi) *For the triangle* $X_kY_kZ_k$, we can show that $B'C'$ and E_kF_k intersect at X_k; $C'A'$ and F_kD_k at Y_k; $A'B'$ and D_kE_k at Z_k. [D_k, E_k, F_k are the points of contact of the ex-circle (centre I_k) with BC, CA, AB respectively.]

(vii) Again, *with* $A'B'C'$ *as the triangle of reference*, for all values of λ, μ, ν for which $\lambda+\mu+\nu=0$, the coordinates of the vertices of the triangle of (3·1) are $(0, \nu, \mu)$, $(\nu, 0, \lambda)$ and $(\mu, \lambda, 0)$. The points P, Q, R are $[a^2, c^2-b^2, b^2-c^2]$, $[c^2-a^2, b^2, a^2-c^2]$, $[b^2-a^2, a^2-b^2, c^2]$ respectively. From these we can show that *each of the five diagonal triangles of this section is in perspective with the pedal triangle* PQR; the pole of perspective will be found to be the point $(a^2/\lambda, b^2/\mu, c^2/\nu)$, which is obviously a point on the nine-points circle of $\triangle ABC$.

Thus

PX_0, QY_0, RZ_0 meet at $w_0 \equiv [a^2/(b^2-c^2)$, etc.];

PX, QY, RZ meet at $w \equiv [a^2/(b-c), b^2/(c-a), c^2/(a-b)]$;

PX_k, QY_k, RZ_k meet at w_k, where w_1 is obtained by changing a into $(-a)$ in the coordinates of w; and so on.

The quadrilateral $ww_1w_2w_3$ is inscribed in the nine-points circle of $\triangle ABC$, and its diagonal points are

$$[0,\ b^2/(c^2-a^2),\ c^2/(a^2-b^2)],$$

$$[a^2/(b^2-c^2),\ 0,\ c^2/(a^2-b^2)]$$

and

$$[a^2/(b^2-c^2),\ b^2/(c^2-a^2),\ 0].$$

It will thus be noted that the diagonal-point triangle of $ww_1w_2w_3$ is in perspective with $\triangle A'B'C'$, the pole of perspective being w_0.

It may be left as an exercise for the pupils to verify that w_0 is a Feuerbach point of the triangle of which PQR is the medial triangle. [*Hint*: Take $\triangle PQR$ for triangle of reference.]

5. *Conics touching four straight lines.*

The following important properties of a system of four-tangent conics are well known:

(1) The *centre locus* is the straight line joining the middle points of the three diagonals of the complete quadrilateral formed by the four common tangents;

(2) The *director circles* form a coaxal system, of which the circles on the three diagonals as diameters are members, the radical axis being the directrix of the unique parabola of the system;

(3) The diagonal triangle of the complete quadrilateral *coincides with the diagonal-point triangle* of the quadrilateral whose vertices are the points in which the lines touch a conic of the system (cf. Salmon's *Conic Sections*, § 263);

(4) The diagonal triangle is *self-polar* with respect to every conic of the system.

Direct application of the above general theorems to each of the systems of conics touching the sides of the quadrilaterals

$$(tt_1t_2t_3),\ (abct_0),\ (abct),\ (abct_1),\ (abct_2),\ (abct_3)$$

shows at once that :

(i) The *centre loci* are the straight lines GN (the Euler line of $\triangle ABC$), GK, GI, GI_1, GI_2, GI_3 respectively.

(ii) The five diagonal triangles of § 4 are all self-polar with respect to the Steiner ellipse of $\triangle ABC$.

(iii) *The nine-points circle, the in-circle, and the three ex-circles of* $\triangle ABC$ *are the polar circles of the five diagonal triangles, which are therefore obtuse-angled triangles having their orthocentres at the centres of those circles.*

(iv) The triangle $X_0Y_0Z_0$ is the common self-conjugate triangle of the Steiner ellipse and the pedal ellipse of $\triangle ABC$.

(v) The triangle $X_0Y_0Z_0$ is the diagonal-point triangle of each of the quadrilaterals $\pi\pi_1\pi_2\pi_3$, $\xi\xi_1\xi_2\xi_3$, $A'B'C'w'$, $PQRw_0$. Hence :

$\pi\pi_1$, $\pi_2\pi_3$, $\xi\xi_1$, $\xi_2\xi_3$, $B'C'$, $A'w'$, QR and Pw_0 meet at X_0 ;

$\pi\pi_2$, $\pi_3\pi_1$, $\xi\xi_2$, $\xi_3\xi_1$, $C'A'$, $B'w'$, RP and Qw_0 meet at Y_0 ;

$\pi\pi_3$, $\pi_1\pi_2$, $\xi\xi_3$, $\xi_1\xi_2$, $A'B'$, $C'w'$, PQ and Rw_0 meet at Z_0.

(vi) The triangle XYZ is the diagonal-point triangle of each of the quadrilaterals $A'B'C'\xi$ and $DEF\pi$. Hence :

$B'C'$, $A'\xi$, EF, $D\pi$ meet at X ;

$C'A'$, $B'\xi$, FD, $E\pi$ meet at Y ;

$A'B'$, $C'\xi$, DE, $F\pi$ meet at Z.

(vii) For the triangle $X_kY_kZ_k$, replace ξ, π by ξ_k, π_k respectively in (vi).

[It should be remarked that some of the intersections of (v), (vi) and (vii) have already been noted in § 4.]

6. *Two well-known theorems on director circles.*

Advanced pupils should be familiar—if they are not, they should become familiar—with the following theorems :

Theorem I : The polar circle of a triangle cuts orthogonally the director circles of all in-conics of the triangle.

Theorem II (*Gaskin's theorem*) : If a triangle is self-polar with respect to a conic, its circumcircle cuts orthogonally the director circle of the conic.

An immediate deduction from the latter theorem is the following :

Theorem III : The circumcircle of the diagonal triangle of a complete quadrilateral cuts orthogonally every member of the coaxal system of director circles of the system of conics touching the sides of the quadrilateral.

I have already communicated to the *Gazette* analytical proofs of Theorems I and II, making use of my new form of the equation of the director circle of the conic whose areal *envelope* equation is $ul^2+vm^2+wn^2+2fmn+2gnl+2hlm=0$. As the form is required in the next section, it may be given here for ready reference. The director circle is

$$uD_{AA}+vD_{BB}+wD_{CC}+2fD_{BC}+2gD_{CA}+2hD_{AB}=0, \quad \ldots\ldots(6\cdot1)$$

where D_{AA} is the power of the point (x, y, z) with respect to the point circle at the vertex A of the triangle of reference, and D_{BC} is the power of the point with respect to the circle on BC as diameter.

7. *Application of the theorems of* § 6.

(1) From *Theorem I*, the polar circle of $\triangle ABC$ cuts orthogonally the director circles of the Steiner ellipse, the pedal ellipse, the in-circle, and the three ex-circles.

(2) From *Theorem III*, we see that the circumcircles of the five diagonal triangles of § 4 are members of a *net* of circles *which have the director circle of the Steiner ellipse for their common radical circle*. By (1) above, the polar circle of $\triangle ABC$ is also a member of the same "net". The envelope equation of the Steiner ellipse is $\Sigma mn=0$ (*vide* 2·1), and hence from (6·1), the equation of its director circle is

$$D_{BC}+D_{CA}+D_{AB}=0,$$

which reduces to

$$\tfrac{1}{3}(\Sigma bcx\cos A)(\Sigma x)-\Sigma a^2yz=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(7\cdot1)$$

showing that the director circle of the Steiner ellipse, the circum-circle, the nine-points circle and the polar circle of $\triangle ABC$ are coaxal (*vide* § 4 (iv).)

(3) Let U_0, U, U_k be the circumcentres of the triangles $X_0Y_0Z_0$, XYZ, $X_kY_kZ_k$ of § 4. Now since the circle $X_0Y_0Z_0$ and the polar circle of ABC both cut orthogonally the director circles of the Steiner ellipse and the pedal ellipse, GK is the radical axis of the first pair of circles and HU_0 of the second pair. Hence (and similarly) we see that the centres U_0, U, U_k lie on the straight lines through H at right angles to GK, GI, GI_k respectively. The lines HU_0, HU_1, HU_k are the directrices of the parabolas which touch the sides of the quadrilaterals $(abct_0)$, $(abct)$, $(abct_k)$ respectively.

(4) Again let us consider the three following circles:

(i) The polar circle of $\triangle ABC$, (ii) the polar circle of *any one* of the diagonal triangles of § 4 (*vide* § 5 (4) (iii)), and (iii) the circumcircle of *that* diagonal triangle. Since their centres are not collinear, the radical axes of the circles taken in pairs meet at their radical centre which for the triangles $X_0Y_0Z_0$, XYZ, $X_kY_kZ_k$ lies on GK, GI, GI_k respectively.

(5) Again, the circle $X_0Y_0Z_0$ is the common radical circle of the director circles of the nine-points circle, the Steiner ellipse and the pedal ellipse of $\triangle ABC$. Hence U_0 must lie on the radical axis of the

director circles of the nine-points circle and the Steiner ellipse. The equation of the latter director circle has already been found in (7·1). The director circle of the nine-points circle may be found as follows :

The centre of the director circle is N and the square of its radius is $\frac{1}{2}R^2$. Hence from (1·1) its equation (with respect to $\triangle ABC$) is

$$[\Sigma(AN^2 - R^2/2)x](\Sigma x) - \Sigma a^2yz = 0.$$

But $AN^2 - R^2/4$ is the power of A with respect to the nine-points circle, and hence equal to $\frac{1}{2}bc \cos A$. Therefore, the equation of the director circle becomes

$$\tfrac{1}{2}(\Sigma bcx \cos A)(\Sigma x)^2 - \tfrac{1}{4}R^2(x+y+z)^2 - \Sigma a^2yz = 0. \quad \ldots\ldots\ldots(7\cdot2)$$

Hence from (7·1) and (7·2), the radical axis of the two director circles is found to be

$$\tfrac{3}{2}(\Sigma bcx \cos A)(\Sigma x) - R^2(x+y+z) = 0. \quad \ldots\ldots\ldots\ldots(7\cdot3)$$

This line, as has already been pointed out earlier, is at right angles to GN (the Euler line of $\triangle ABC$). If the line cuts the Euler line at T, what is the ratio OT/TG, where O is circumcentre of $\triangle ABC$? Substituting the absolute coordinates of O and G in the left-hand side of equation (7·3), we find that $OT/TG = 3$. *Hence T is the middle point of ON, and the line* (7·3) *is the perpendicular bisector of ON.*

The point U_0 is therefore the intersection of the perpendicular bisector of ON and the line through H at right angles to GK. We thus have a geometrical construction for U_0.

I now leave it to the readers to " carry on ".

F. H. V. GULASEKHARAM.

1314. In some cases it is impossible to measure the entire parent population and in other cases it is impracticable to do so. Suppose a physician was interested in the blood pressure of American men between thirty and thirty-one years of age. He could never expect to get complete data for all the men in the parent population. Not only would it be impossible ; it would be unnecessary, expensive, and a waste of time and energy. An excellent judgement could be made by the study of a properly selected sample.—C. H. Richardson, *An Introduction to Statistical Analysis*, New York, 1935, p. 15. [Per Mr. E. S. Pondiczery.]

1315. Yet it is certainly something of an accident historically that this particular discipline (Geometry) should have become fashionable in the Greek Universities, and later embodied in the curricula of secondary education. It would be difficult to overstate how much the liberty of human thought has owed to this fortunate circumstance. Since Euclid's time there have been very long periods during which the right of unfettered individual judgement has been successfully denied in legal, moral, and historical questions, but in which it has, none the less, survived, so far as purely deductive reasoning is concerned, within the shelter of apparently harmless mathematical studies. —R. A. Fisher, *The Design of Experiments*, 1937. [Per Mr. R. A. Fairthorne].

THE TEACHING OF ELEMENTARY INEQUALITIES.

By Joyce Linfoot.

In many School Certificate courses of Algebra inequalities are scarcely mentioned, and yet in Higher School Certificate work it is often assumed that the pupil can, without any further teaching, carry out easy manipulations of inequalities. It is true that the gap is a small one, but even the most intelligent pupils may fail to bridge it satisfactorily for themselves, and difficulties will then arise. One example of such a difficulty is noted in the Report of the Mathematical Association on the Teaching of Mechanics in the section on friction. The Report reads: "There is a tendency among some pupils to proceed at once to the limiting case when equilibrium is about to be broken, to write down the equations for this case, manipulate them, and finally change the resulting equality into an inequality, using $>$ or $<$, according to the answer required."

If in any particular case we try to track down the reason for such a tendency, we usually find inadequate teaching. Students (sometimes even University students) often have no clear-cut knowledge of the following elementary facts:

I. If $A>B$, then $A+C>B+C$ and $A-C>B-C$;

II. If C is positive and $A>B$, then $CA>CB$;

III. If C is negative and $A>B$, then $CA<CB$;

where A, B may be either positive or negative. They have never been taught these properties, and have lacked the initiative to formulate them for themselves.

Is there any reason why this information should not be included in the School Certificate course? The idea of inequality is not a difficult one for a child to grasp. On the contrary, a child is likely to use and understand the words: "I am bigger than Harry" long before he says "I am the same size as John". . . . if only because ordinary brothers and sisters are so much commoner than twins. It is equality which is the more sophisticated idea! Of course this primitive concept of inequality only enables its possessor to deal with positive quantities. Properties I and II scarcely need more than stating when all the numbers concerned are positive; it may be this fact which leads to underestimation of the teaching necessary before the properties in their complete form can be appreciated.

The first difficulty encountered is that the most natural way of extending the idea of inequality to negative numbers would seem to be to say, not $-4>-5$, but $|-5|>|-4|$. (I use this notation for simplicity.) To teach the usual extension to negative numbers by the statement that $A>B$ if $A-B$ is positive is unenlightening. It is not until a boy is familiar with the representation of numbers by points on a straight line, and until he can replace the symbol "$>$" by the mental picture "to the right of" that he is likely to have any confidence in dealing with such inequalities. In terms of

this representation the three properties become extremely simple. They can be stated as follows :

I. If A is to the right of B, and both points are moved the same distance in the same direction, the new position of A is to the right of the new position of B.

II. Multiplication by a positive quantity (which corresponds to a simple magnification) does not affect the order of points on the line.

III. Multiplication by a negative quantity reverses the order of points on the line.

Each of these statements is capable of a simple practical demonstration. The work would only be a small addition to the usual course on directed numbers. It might not be possible to include it when these were first introduced, but for revision it would be most valuable material. Work on simple equations could then be followed by similar work with inequalities.

Another field in which much greater use could be made of inequalities is that of elementary geometry. This is recognised in the recent Report on the teaching of Geometry ; but the Report refers only to an inequality group of theorems. In point of fact, almost every geometrical theorem can be put in the form of a statement about inequalities, and many theorems gain in clarity when they are so expressed. The following is an obvious example :

If A, B are two given points, and P, C are two points on the same side of AB, then

if P is outside the circle ACB, $\widehat{APB} < \widehat{ACB}$,

,, on ,, ,, $\widehat{APB} = \widehat{ACB}$,

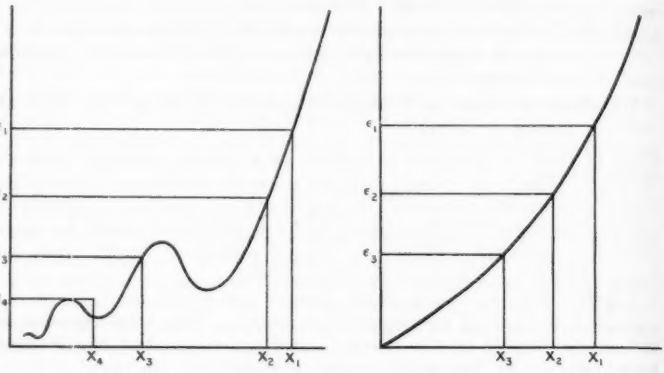
,, inside ,, ,, $\widehat{APB} > \widehat{ACB}$.

Each of these three statements is capable of a simple, direct proof, and taken together they imply the truth of all the corresponding converse statements.

In general, if we use inequalities in this way, we can avoid many *reductio ad absurdum* arguments by using the fundamental one that, of two quantities, one must be either greater than, equal to, or less than the other, and that if two of these contingencies do not hold, then the third must.

The effect of such teaching is to establish the concept of a locus as a dividing line, on either side of which opposite inequalities hold. This immediately makes clear the dual aspect of a locus, that any point on the locus satisfies the condition, and that any point satisfying the condition is on the locus. In Sixth Form analytical geometry the same question arises when we consider whether a given point is inside or outside a given curve, or on one side or the other of a given line. The line $ax+by+c=0$, for example, divides the plane into two parts, in one of which $ax+by+c>0$, and in the

other of which $ax+by+c<0$. The sign of c immediately decides in which of these two parts the origin lies. Hence if $ax'+by'+c$ has the same sign as c, (x', y') and the origin lie on the same side of the line.



A more important way in which early familiarity with inequalities can be of use to the Sixth Former is in enabling him to understand the concept of a limit. Let us consider one of the first examples he will meet, $\lim 1/n$ as $n\to\infty$. The essential idea, of course, is that we can ensure that $1/n$ is less than any positive number (chosen as small as we please) by taking n greater than some other number. There is, in fact, no real escape from "epsilonology": "Given $\epsilon>0$, $\exists N$, $1/n<\epsilon$ for all $n\geqslant N$". But since this notation is unsuitable for elementary teaching, we must work by diagrams and by numerical examples. For instance, in the case of $1/n$, we say to the pupil, "How near to 0 would you like $1/n$ to be?" Suppose the answer is, "Within ·000,000,001". Then all we have to say is, "$1/n<$·000,000,001 if $n>1{,}000{,}000{,}000$". The point which the student most often fails to grasp is, I think, that

$$\text{for } \textit{all } n>1{,}000{,}000{,}000, \quad 1/n<\cdot 000{,}000{,}001,$$

and this failure is due to unfamiliarity with inequalities. He mentally replaces the inequality argument by something like this:

"When $n=100$, $1/n=\cdot 01$,
$n=1{,}000$, $1/n=\cdot 001$,
$n=10{,}000$, $1/n=\cdot 0001$, and so on.

Then as n becomes bigger and bigger, $1/n$ becomes smaller and smaller, *i.e.* $\lim_{n\to\infty} 1/n=0$".

For monotonic functions he finds this line of argument works well enough, and he believes himself to have an adequate conception of

the idea of " limit ". The existence of such limits as $\lim_{x \to 0} x \sin 1/x$ is then a grave difficulty, and involves re-establishing his ideas on a correct basis. It would be sounder teaching if we could make it clear from the start that it is not only monotonic functions which can tend to a limit. The diagrams opposite illustrate the point which may be missed by the young Sixth Former.

To sum up, the difficulty in teaching limits is that the student tends to concentrate on the value of the function at each of a series of values of the variable, that is, on a series of equalities, whereas what he should be considering is the whole range of values of the function for a certain interval of values of the variable, that is, an inequality. J. L.

1316. In the general plan of this work, and in several particulars, I have deviated from some of the methods often made use of, partly in attempting to put the subject in a simpler and clearer point of view, and partly in avoiding certain steps of reasoning which appear to be defective. One of these is the fallacy of establishing premises on a certain implied condition, and drawing a conclusion from them by a direct violation of that condition. An example of this is to be found in a proof often given of the principle of indeterminate coefficients, in which the factor x is divided out of the equation

$$Bx + Cx^2 + Dx^3 + \&c. \ldots = 0,$$

which of course tacitly assumes the condition that x is not zero; in this manner is obtained the equation

$$B + Cx + Dx^3 + \&c. \ldots = 0;$$

and then by putting $x=0$, contrary to the implied condition, the conclusion $B=0$ is arrived at. . . .

The *assumption*, that $f(x+h)$ can be expanded in a series of the form

$$A + Bh^\alpha + Ch^\beta \ \&c. \ldots$$

seems to me to be a serious defect in the common method of establishing Taylor's Series, and thereupon the principles of the Differential Calculus. This assumption is usually justified by arguing, that if we find definite values for A, B, C, &c. it shews that the assumption is correct. Now this argument may be stated thus:

"If the assumption that $f(x+h)=A+Bh^\alpha+Ch^\beta+\&c.$ be true, then A, B, C, &c., must have definite values. But we can in general obtain definite values for A, B, C, &c. ... (*e.g.* by the method of indeterminate coefficients). Therefore the assumption is true."

This is clearly a fallacious argument, for to warrant the conclusion the first premise should have been this: "If the assumption *be not true*, definite values cannot be obtained for A, B, C, &c."

These defective steps of reasoning and others which might be mentioned, are objectionable, not because they lead to erroneous conclusions, but because they ought not to be found in a subject like the present, in which every thing should be conformable to the strictest rules of logical deduction.—Rev. M. O'Brien, *Elementary treatise on the differential calculus* (Cambridge, 1842), p. vi-viii.

The interest of this Gleaning lies in the date, which seems to oppose the usual belief that rigorous demonstrations in the calculus were unknown in nineteenth-century Cambridge. [Per Prof. E. H. Neville.]

SOME DOUBTS DOUBTED: A REPLY TO MR. BLACKWELL.

By C. O. Tuckey.

Mr. A. Blackwell's doubts, published in the February *Gazette*, make sad reading for one who has spent his life teaching mathematics.

Mr. Blackwell and similar pessimists attack the basis of our ordinary curriculum; but he differs from some gloomy writers by including certain constructive suggestions which need consideration.

In defending the present curriculum I would wish to submit that the attacks upon it are based upon a series of postulates of which some are of doubtful validity and the rest untrue.

Further, the suggestions made are in many cases those which, without Mr. Blackwell knowing it, the Mathematical Association has for many years been urging in its reports, while of the others some are to a considerable extent self-contradictory, though some of them may be of value.

Consider first some of the postulates on which the condemnation of our present curriculum appears to be based.

(i) "A scientific book is said to be 'popular' if it is non-mathematical." This is a half-truth. A scientific book is said to be "popular" only if it is easy to read; that is, if it *demands no special knowledge* on the part of the reader. Now it so happens that science depends upon mathematics; thus the special knowledge which must be skirted round, suggested, fudged and dodged is knowledge of mathematics. If, however, science depended on, say, law or Latin or German or physical geography, it would be these subjects whose absence would be necessary before the book could be "popular". Thus the correlation of "popular" with "non-mathematical" is accidental and in a way a compliment to mathematics. It is not an indication that the man in the street hates mathematics. On the contrary the fact that these books about exciting new discoveries are so much easier to understand if one does possess some mathematical knowledge must often cause "the man in the street" to regret his wasted opportunities.

(ii) "Pupils have a positive distaste for the subject of mathematics while they all love all other subjects at which they are expected to work hard."

Or even "children have less interest in mathematics than they have in other subjects of the curriculum".

This is untrue. Pupils tend to enjoy (*a*) those subjects which they personally find easiest; thus a boy in a top mathematical division and a bottom French one will tend to enjoy mathematics more than French and *vice versa*. (*b*) Those subjects which are best taught. When they are opposed, (*b*) probably in most cases overrides (*a*). As we had it drummed into us when we started teaching, "if the boys don't enjoy the work, it is *your fault*". Really bad teaching

will kill the interest of a class in any subject, and really brilliant teaching will make any subject interesting. The intermediate kind of skill to which most of us can attain (if we don't say "it is the fault of the curriculum" and give up trying) should suffice to make mathematics interesting to all normal pupils. Perhaps for some types of experimental science less skill is needed and for Latin grammar somewhat more, but here I go *ultra crepidam*.

(iii) "The normal healthy child never stops in the middle of a game of football or hockey to ask, 'what is the good of doing this'. They love it far too much."

This is untrue of *all compulsory games*; more so no doubt of some than of others. Would Mr. Blackwell say of cricket or athletics what he says of football or hockey?

(iv) "The welter of figures in which we live—cooking recipes, railway time-tables, etc.",

(*a*) do not appear in our mathematical books;

(*b*) are easy to gather together into a connected subject;

(*c*) are of general interest as opposed to each one interesting some special persons;

(*d*) are a possible subject of teaching at the early age suggested.

The answers here are (*a*) they *do* appear, (*b*) very difficult, (*c*) doubtful, (*d*) very doubtful.

Mr. Blackwell has grave doubts as to contacts between the mathematics which we teach and real life, and he asks for an instance in which the theorem that the sum of the angles of a triangle is two right angles touches real life. It is a little hard to know what he or anyone else means by "touches real life", but I suppose that he would allow that the making of maps, so much used by motorists, cyclists, hikers and others, touches real life. I do not claim to be an expert on the making of maps, but I suppose it to be fundamental that a base line AB is measured, and that to find the distance of a point C, say an inaccessible peak, from the points A and B, the angles BAC and ABC are measured and CA, CB found by the sine formula. It is to me a pleasing subject of speculation to consider whether Mr. Blackwell, when attempting to climb the inaccessible peak of our dreams, complete with theodolite, etc., in order to measure accurately the third angle of the triangle, will continue to consider that the knowledge which would have saved him the climb has no bearing on real life.

Some of Mr. Blackwell's suggestions for reform will now be considered. Of these one is of recent date and needs careful thought. He describes in detail the devising of the subject called "general science" and suggests that a similar method should be adopted to devise "general mathematics". It is quite possible that the method adopted by the scientists has advantages over the methods of the Teaching Committees of the Mathematical Association whose efforts and reports Mr. Blackwell so studiously ignores. At any rate

it might well be worth trying and it would be interesting to see what sort of curriculum would be produced by this method.

It will, however, be a few years before we can get a considered opinion as to the success of " general science " as a subject.

Mr. Blackwell is also a great admirer of Mr. Hogben. Here I think many of us will agree with him—in fact most of us have been trying, for years before Mr. Hogben's book appeared, to be ready with the appropriate historical allusion or real-life-connection to add interest to the work. Unfortunately few of us have Mr. Hogben's varied knowledge or his mordant wit. Unfortunately also Mr. Hogben's book, like Euclid's, is primarily for adults and needs a fair amount of general knowledge if it is to be fully appreciated.

If Mr. Blackwell will read the reports issued of recent years by the Mathematical Association he will find instance after instance where these reports have prophetically plagiarised his advice to try to bring mathematics into contact with real life, to connect it with other subjects of the curriculum, to enrich it with historical allusions, to reduce drill examples to a minimum, and so on.

When, however, it comes to his other suggestions for improving the curriculum Mr. Blackwell finds it less easy to avoid self-contradiction. For instance he makes these suggestions :

" The foundations of a child's study can be really broad and comprehensive."

It is possible to give children " a complete wide view of a subject, showing how it affects life at as many points as possible ".

It is possible for a child to possess " a growing nucleus of exact knowledge which it thoroughly understands and can use with ease ".

Of these sentences the third seems to suggest the exactly opposite course to the other two ; at any rate it might well be adopted as a slogan by those who believe in constant drill in multiplication of decimals.

And what of this suggestion? " To start with, the curriculum must be complete in itself. There must be no ' loose ends ' whose significance is only fully appreciated by further work. . . ."

This seems to me like saying, " the year 1940 must be complete in itself. . . ." Alas that some thought this of the treaty of Versailles!

It is difficult also to let pass the remarks on surds and on text-books with innumerable examples. Mr. Blackwell's " dead cert. question for School Certificate, viz. surds ", leaves, me more than doubtful as to the " cert.". I suppose I have set more examination papers than most people and I do not plead guilty to setting many surds.

It is a pity that he misunderstands the idea of a book with far more examples than appear to be wanted. The idea is that a boy revising in 1941 need not do the same examples that he did in 1940.

I think one always oscillates from liking short books with just about the right number of examples to liking long books with far

too many, and then back again. The crab of the first kind is that one gets sick of them in a few years, and of the second that one can never get the boys to feel that they have mastered a particular part of the subject. Just now, having reviewed *Mathematics for the Million* and read most of *Anthony Adverse*, my liking for very long books has gone with the wind ; but I should not think it fair to suggest that even half the examples contained in the longer type of book are expected to be done by all pupils.

As to the surd example which Mr. Blackwell pillories,

$$\frac{(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})}{(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{3}-\sqrt{2}+\sqrt{5})},$$

it is of course one that should only be used in special circumstances, but it is tempting to try and defend it and such like by the factor-of-safety idea. May I illustrate the idea thus? Jack has a job to feed daily some dangerous animals who are kept in by a ditch 10 ft. wide. At times, say once a week, the animals turn nasty and Jack has to run for his life and jump the ditch for safety. Would you suppose that Jack will spend some of his spare time in improving his long jumping? You would. Will Jack be satisfied if his best jump is 10 ft. 6 in.? Not a bit of it. He will practise till his best jump is 15 ft. or so and only then will he feel at all confident of clearing the 10 ft. in the hour of need.

Even so with " sin 45°, sec 30°, $\frac{1}{\sqrt{2}}$, $\frac{2}{\sqrt{3}}$ to be got *easily* ". They will never be got easily if they remain the most difficult thing the boy has done with surds. He must go at least a little farther. Probably he will not go as far as the example above, but if he can do that one, he can not only handle surds fairly easily, but he can use the results for $(a+b)(a-b)$ and $(a \pm b)^2$ in rather difficult cases.

As an instance of the special circumstances in which such unnecessarily complicated examples are of value I would suggest this : the chapter on surds has been sufficiently dealt with and the next chapter is due, but a third of the class are found to be absent owing to a drill competition or an influenza scare, so it seems unwise to start the new subject. Here is where the discussion of such an example as that pilloried will provide valuable revision without causing the rump of the class to feel that they have been put back to what they have done before. Note also that while the class think they are doing surds they are really revising $(a+b)(a-b)$ and $(a \pm b)^2$ for which it is hard to devise sufficient revision without monotony.

This, I suggest, is why such examples continue to appear in our more comprehensive textbooks.

Finally, as to Mr. Blackwell's suggestion that " the present syllabus is framed to produce what we might call ' professional mathematicians '—at any rate it is laid down with a view to the requirements of future VI Form mathematical specialists ".

This may or may not have been the view of some who in the past have framed examination syllabuses. It is *emphatically not the view of the Mathematical Association* as expressed year after year by the reports of its Teaching Committees.

As regards the dreaded certificate examinations, Mr. Blackwell and others will be well advised to take their courage in both hands and try to make the mathematics they teach as interesting as they can and to teach as much mathematics as the time available allows without always looking to see what is a "dead cert." in the papers. The bulk of their pupils will then take these examinations in their stride, and the residue that have to be crammed up for a term or two before the examination will have far more chance than if the cramming method had been the normal method of teaching.

Let the doubtful teacher teach *mathematics* and certificates for his pupils, with credits and perhaps even distinctions, will be added unto him.

C. O. TUCKEY.

1317. A SQUARE DEAL FOR THE NAVIGATOR.

If a flying boat is forced down at sea and its position is known to three miles, seven square miles must be searched for it; if, however, its position is only known to within fifty miles—and I think pilots will agree that, until recently, this was considered very good accuracy on long flights—no less than 7857 square miles of ocean must be searched. Nor is this all, if rescue is delayed and the rate of drift is not known, the circle of uncertainty becomes a cylinder; and finally, because the wind may have changed, a cone.—F. Chichester, *Flight*, 2nd March, 1939. [Per Mr. R. A. Fairthorne.]

1318. As taught today the subject is introduced to the student somewhat as follows: "This is Mr. Cartesian Geometry who can do almost everything that Euclid did. He has a panacea for all geometrical troubles, which like all panaceas will sometimes act quickly and at others with alarming slowness, if at all. He has a reputation for dealing with loci, though often the net result of his treatment is a complicated equation which leaves you puzzled as to what it could possibly signify."

How much better would it be if they said instead: "This is the famous Magician Cartesian Geometry. He has a Magic Carpet which he calls 'Z' which will carry you to the Land of the Far Away where you will see for yourself those rebellious parallels which would not meet however far they were produced by Euclid behaving quite as meekly as any other pair of lines. And he has another Carpet called 'i' which will carry you to the Land of the Unseen, a Ghostly Land wherein are to be found the spirits of a pair of points or lines when they suddenly disappear locked in each others arms. And if you should spread both the carpets, why, who can prevent you from reaching even up to those miraculous twin shrines which stand right at the very borders of the Metrical Domain! You are indeed lucky to have Cartesian for your travelling companion."—A. Narasinga Rao, *The Mathematics Student*, vol. v, p. 182.

1319. He then went on to say that the latest fashion in the United States is the "finger free" glove. Briefly speaking, it is a glove which has length, width, and thickness instead of merely length and width. That is to say, there is a little gusset on the outside seam which extends up to the finger tips instead of tapering to a point there. In other words, it is a three-dimensional glove.—*Daily Mail*, January 27, 1939. [Per Prof. L. M. Milne-Thomson.]

CORRELATION OF THE TEACHING OF GEOGRAPHY AND MATHEMATICS.*

By J. Fairgrieve.

There is a healthy attempt to break down the barriers that have hitherto existed between the subjects that are taught in schools. At the same time it must, I think, be taken as an axiom that the several major subjects of the curriculum, especially in secondary schools, should be organised without reference to each other if each is to make use of its own distinctive discipline. A complicated articulation would be unworkable.

This does not prevent adjustments which may improve the teaching of two subjects that have relations with each other, as is the case with mathematics and geography. But it seems to imply that the correlation of the teaching of mathematics and geography, like the correlation of the teaching of any other school subjects, is primarily a matter for the common room rather than the syllabus ; it need not, however, be any the less close for that reason.

Obviously there is much in mathematics with which geography has no concern, there is much in the geography course on which mathematics throws no light, but there is some material which while it belongs to one subject yet is better understood if related material in the other has been, as we say in teaching jargon, " done ". Also there is material which is more or less common and it is desirable to see in which department or by which master or mistress it should be introduced and carried on.

Now I know that in speaking to this audience I ought, if I were sensible, to confine myself to speaking of those ways in which the teaching of mathematics can be correlated with the teaching of geography because you are not concerned with the ways in which the teaching of geography can be correlated with the teaching of mathematics, but I find it difficult to separate the two things, and I am going to shield myself, if I may mix my metaphors, behind my terms of reference which were without distinction to speak on " the correlation of the teaching of geography and mathematics ".

First I shall deal with some fundamentals and state what seems to me to be probably the chief contribution that the teaching of mathematics makes to the teaching of geography. I was a mathematician before I was a geographer ; in fact my paper qualifications for teaching are almost entirely mathematical and scarcely at all geographical. It is true that I have forgotten a marvellous deal of mathematics, but though I have forgotten a deal, and perhaps *because* I have forgotten so much, I see more clearly than ever the important things that the learning of mathematics and the teaching

* A paper read to the London Branch of the Mathematical Association.

Both in content and form this address was written to be listened to, not to be published, and the author hopes that what has been described as " a blend of insult, urbanity and instruction " will be read in the spirit in which it was written and that readers will benefit from the instruction and pardon the insults.

of mathematics taught me ; it is indeed the stressing of what I have learned from mathematics that is the special contribution that I have made to the teaching of geography.

And one of the important things that I have learned is this. If mathematics has been well taught there should have grown up in the minds of the learners the fundamental assumption that you can come to valid conclusions only when you use all your relevant data and that you cannot come to valid conclusions if you have not all the relevant data in hand. " Assumption " is not a very good word, but I wish to stress the fact that what is wanted is more than knowledge, more than belief, more even than conviction. I call it an assumption because I wish to imply that it is so deep-seated that you do not consciously think about it at all ; you take it for granted.

Now, you probably do not see how important this is, so let me give an example—being mathematicians you believe in examples. Geographers have, sometimes, a way of saying things like this: " The valley of the Kennet is easier to pass along than are the hills to north and south, so a road runs along it, and so does a railway ; another road and another railway follow the other valley, that of the Thames eastward from Goring ; the place where they meet is important so—they sometimes say ' therefore ', instead of ' so ', but it means the same—so Reading has sprung up." Or again, " Lancashire has a coalfield, it faces America whence cotton comes, also being on the west it has rain and a damp atmosphere, therefore cotton is manufactured." Or, " A stream wears away the land and brings down silt which sinks in still water and is deposited on the bottom, therefore where there is an absence of tides a delta is formed."

Now, if you—or we—mathematicians had really taught mathematics, or rather if our pupils had really learned the mathematics that we thought we had taught them, *i.e.* if they took for granted that they could not come to valid conclusions if they had not the relevant data in hand, they would put their fingers to their noses at such a geography master and say, " Rats, what about your relevant data? " or words to that effect.

Consider the delta. What is said is correct till you come to the conclusion ; a stream does bring down silt, it does deposit it in still water where there are no tides, but how does that explain the fact that the surface of the delta is *above* the surface of the water? Silt does not sink *upwards* out of the water altogether. There is obviously something wrong ; some of the relevant data have not been used. What has not been used—and it is rather symptomatic of our education—is biology, in this case botany. A floating stick or branch is caught in the shallow water and in turn catches grasses and leaves, aquatic vegetation spreads from the shores and continually dying and decaying becomes soil which in turn is material for further growth. An occasional flood may provide silt and aid the process till the ground is dry enough for other forms of vegetation

to grow in their turn. I watched a small delta in a pond forming for a dozen years in this way, and I know.

I stress this because, as I have said, it is symptomatic of our education to omit biology. " Science " has meant physical science, and, backed by mathematics has resulted in an attitude of mind that thinks that results always inevitably follow from definite data, and they do not where life is concerned. Science which takes no account of biology is not scientific, as it misses out half the data. It is our job to see that all the relevant data are used, even if it means that we then see that we have not enough data and cannot come to definite conclusions, that there are, for example, two *alternative* solutions to a quadratic equation ; *either* $x=1$ *or* $x=2$, and *we do not know which is true or if both are true.* Do we stress enough that they are possible alternatives?

And what of Lancashire? It is certainly not true that " therefore there is a cotton industry in Lancashire " ; South Wales " faces west and is opposite America whence cotton comes, also being in the west it has rain and a damp atmosphere, and it has a coalfield ", yet not an ounce of cotton is manufactured in South Wales. That spoils the " therefore " for cotton in Lancashire, or the insidious " so ".

And Reading? There are many valleys without roads running along them and there are many places where roads meet and there are no towns. Data absent again. The fact is that where men are concerned there is almost always an " unknown " which forbids logical deductions. Man chooses sometimes to do the easy thing, but he sometimes chooses to do the difficult thing. Water always chooses the easy way downhill, not so man. He chooses to go to the South Pole and other places difficult to reach. He sometimes, also, chooses to do just what his father or neighbour does, or he does it a little differently. You will never explain the Lancashire cotton industry in terms of logic, and if people try to do so it is because their mathematical teachers failed at school to teach them one of the chief lessons of mathematics, that your conclusions are limited by your data (and very much limited when man is one of the data). In the case of Lancashire you can learn only how, not why, the cotton industry grew (biology again).

Does all this seem too philosophical and academic? For myself I do not think it is. My friends call me a realist and I have even been called a functionalist, and I plead guilty to being both if they mean that I try to face realities and that I desire to know what we teach things *for*. I think that the matters of which I have been speaking are important because they raise the question of what you really teach mathematics in school *for* ; what in fact do you expect to have left when your boys and girls have forgotten all the mathematics they have learned?

Anyhow, so much for that. I have spoken of what geographers may learn from mathematicians. What can mathematicians learn from geographers? There is that simple and straightforward subject

of arithmetic. Yet I wonder if it is so simple and straightforward as it looks. What, for example, has geography to say to the simple statement that $2+2=4$? Simply, I think, that it is not always true, and if it is not always true then the mathematical heavens seem to fall. I do not mean to say that it is not true in some highly abstract state of which I know nothing, but—I said that I was a realist—that it is not true on this world—I am also a geographer—as we know it. Another example: "Add two salmon in the Tweed and two salmon in the Yukon." Do you "get" four salmon? By no means. Being a geographer I know that the Tweed and the Yukon are some distance apart and that by the time you added them you would not have four salmon but only a mess and a smell.

You may say that I am only being funny and that I am not dealing with the teaching of mathematics. Nothing is farther from my mind. I am in deadly earnest. This is relativity as geography applied to the teaching of mathematics, and as far as I can gather the application of Einstein's discoveries has not filtered down—I am mixing metaphors again—to the classroom. I use the particular illustration, beg pardon, "example", as a means of suggesting its importance and directing attention to the really important application that arithmetic, like geography, deals with *things* and that whenever either of them forgets this fact it loses its value. Before I can add things I have to know what they are, where they are, "2 what and 2 what, and where are they?"

The real point is this. Youngsters cannot do so-called problems because they do not think of things as they are, real things; they have been trained to think primarily of the operation rather than of the things operated on. Of course they have not been told to learn this; they have learned it *because* they have not been told, *because* it is assumed that it is the operation that matters, and it is what you assume that you teach. Because of this they add when they should subtract, they multiply when they should divide. They *don't* make those mistakes in real life in dealing with things; they may make mistakes in adding or subtracting, but they don't add when they should subtract.

All arithmetic from the beginning, and ever afterwards, should deal primarily with things; then operations would look after themselves; at least if they do not it is no use teaching them. And usually (in fact to be honest and rude I may say "always") it is the other way about. In learning the first four rules there is only one place where it might be argued that theory comes in, and then there is a doubt about it; yet youngsters are continually being taught that 2 and 1 make 3, 2 and 2 make 4, when it is all a question of "2 what? and where are they?" Two boys and two boys add up to four boys, two beans and two beans add up to four beans, but what about two boys and two girls, two runner beans and two broad beans or two lumps of sugar and one cup of tea? If Henry VIII had six wives, how many had Henry II? Bradman made 20 runs in x minutes, how many did he make in $2x$ minutes? Six men built a

wall in ten days, how long will 100 men take? Floors, too, have to be carpeted and walls papered under conditions that never exist in this world.

Now again I do assure you that I am not being funny. I am trying to put my finger on what seems to me to be the great defect of arithmetic teaching to-day. It deals with figures and operations instead of with realities. Nor is it sufficient to work with problems if these are expressed in words. These words are just as much symbols as are figures—idols that stand between you and the realities, images that you worship—and often mean little or nothing. There is no visualisation of the reality. Mathematics deals with unknowns but not that kind of unknown.

Then consider another aspect of the same trouble. A boy in school thinks that he is doing very well if he gets three sums right out of five. Again that is not dealing with realities. In the first place, that is not reckoned as satisfactory anywhere in the world (geography again) except within the four walls of a classroom; neither in business houses, scientific investigations, industry or ordinary household accounts would we tolerate getting only three of our calculations right out of five, or four out of five, or nine out of ten.

In the second place, and even more important, it is just because in the classroom realities are not normally dealt with, that however much it is insisted on, accuracy is never perfect. There is no obvious *reason* why there should be any attempt at accuracy in the arithmetic course except that the master tries to insist on it or the boy thinks that he ought to make at least a decent show.

This is primarily a problem for the mathematical staff and I shall not say much more about it, but there are two matters of correlation of arithmetic with the geographical work that are worth mentioning; one is important, the other may seem of less account.

That which is important, or seems to me to be important, has to do with the word "commerce". It is a word that keeps turning up in all sorts of connections. It turns up in history; it turns up in geography; it is implicit, even when not mentioned by name, in a very great deal of arithmetic. But because it turns up in so many subjects it is no one person's business to deal with it, no one does deal with it and it largely remains without connotations. It is a symbol, an "unknown" like "x" which, as the small boy said, is what we use when we do not know what we are talking about. Here seems to me a matter for correlation. The geographer and the historian may do something, and I suggest that the mathematics master should not only deal with stocks and shares, which remain difficult *because* there is an absence of the necessary background, but that he should concern himself with the actual working of, say, a particular bank or business house as part of the humanising of the subject, a humanising that should also include the arithmetic of citizenship.

The other matter is a minor one, though it has major implications. It has to do with the fact that in geographical work, as in the science laboratory, there are a considerable number of simple calculations to be made; in some cases they have to do with the construction of economic maps. In these, however simple the actual work may be, it is essential, and seen to be essential, that when the calculations are shared by the members of the class, there should be perfect accuracy on the part of everyone if the result is to be valid. There is a *reason* for everyone getting every answer right. That is to say that arithmetic would seem to be taught better in the geography room or the science laboratory than in the arithmetic classroom! I suggest that some of this work ought to be done in the arithmetic classroom. Then among other things pupils may see that contracted methods, instead of being an invention of the devil to plague youth, really do save time.

If you wish to do something more complicated I am sure that geography masters would be glad if your more advanced pupils produced a statistical demonstration of the reasons why the moon has a greater influence in producing tides on the earth's surface than has the sun. I tried it myself with excellent results in the way of interest aroused.

All this suggests an idea that might be worth considering, that arithmetic is a means to an end rather than an end in itself.

Then, to look at the ostensible subject-matter of my address from another point of view, the relations between geography and mathematics and correlations in the teaching of the two subjects are implicit in the first syllable "Ge" of Ge-ometry and Ge-ography. In geometry, the measurement of the earth, there is a vast amount which has directly to do with geography, and some matters about which it is difficult to say in which subject they come in most appropriately for teaching. Among the latter is one about which I wonder whether its importance is realised, "the appreciation of space relations". It is quite distinct from the technical measurement of space and to a large extent is intuitive. It is extraordinarily important in both geometry and geography and yet the cultivation of this intuitive ability to appreciate space relations is much neglected.

In a parenthesis, may I stress the importance of intuition; it is a kind of unconscious or subconscious thinking and reasoning. A poor intellect will do it badly and a good intellect will do it well. To me it seems an even more clear evidence of a superior intelligence to use intuition successfully than to reason consciously about known data accurately. Take the case of the man who has to choose staff to help him in his work, be it business or academic. He may see testimonials and paper qualifications, but the man of affairs sizes up a candidate at an interview without consciously giving due weight to each almost intangible qualification, some of them emotional. Such an estimate of a candidate very often implies the coordination and evaluation of a great amount of data. The man of affairs gets his

resultant without consciously reasoning at all, but that does not make it less intelligent, or less a test of ability.

However that may be, the appreciation of space relations is, at least in part, a matter of intuition ; it is certainly a matter of some kind of feeling or emotion as well as of conscious reasoning, and it can be improved.

In geography it is concerned, for example, with what a mile looks and feels like, realising how far away the north of Scotland, or anywhere else, is from London ; it is concerned among other things with some kind of *realisation* of distances, areas and directions ; it is concerned not merely with a knowledge of the figures which express the numerical values of these things but with what the figures mean. In fact, it boils down to thinking in three dimensions, *i.e.* to thinking solid. This is a side of geography which is often neglected but which is fundamental.

The corresponding side of geometry is also much neglected. We rarely, except in good schools, get as far even as solid geometry ; it is too advanced! Yet there was an Aristotle as well as a Euclid, and it was the former who pointed out that psychologically three dimensions comes first, then two and then one, though logically the opposite order holds ; at least I suppose it holds, though personally I am inclined to agnosticism on the matter. It is certainly true that in drawing, say, isotherms from statistics, very much better results are obtained if one thinks of a line as something lying between two contiguous areas than as something drawn between points ; anyhow that is the idea behind the practice in the meteorological offices of the world. Contour lines and all such lines are determined in the same way. Also an isothermal surface lies *between* masses of air of different temperatures ; in fact it faces both ways. In the same way one should not think only of the surface of a cone and not at all of the surface of the air surrounding it. The real surface, if I may be picturesquely inaccurate, lies between the two.

The fact is that we live—biology again—in three dimensions. A new-born baby at once begins to learn three-dimensional geometry ; put a finger in his palm and his hand will close over the finger ; in a few months he will drop a ball behind him and then look round because he knows where it is ; he has intuitively combined his experiences and already has begun to appreciate space relations. But in school little is done about it. Even when solid geometry is "done" in school the appreciation of space relations, which is something different from ordinary solid geometry, is generally ignored.

In geometrical work it seems to be difficult for teachers to realise that their pupils are not necessarily equally gifted in the intuitional and the logical sides of geometry. Some can appreciate space relations without being able to deal with them logically, and some can deal with them logically without being so able to appreciate them. It is often said, for example, that girls cannot do geometry, and there is indeed some basis for the belief if geometry means

logical geometry. But it will take a good deal more than classroom evidence to make me believe that girls cannot appreciate space relations at least as well as boys. A girl may not be able to prove that the square on the hypotenuse of a right-angled triangle is equal to the squares on the other two sides, and may not see much sense in proving it, but the same girl who " can't do geometry " will make clothes for herself that will *fit*, and if that is not geometry I do not know what it is. I sometimes wonder whether, if the women set the syllabus in geometry, we men would make a very good showing at it. The fact remains that girls and women are continually thinking of dress shapes and space relations in three dimensions in which intuition plays a large if not the whole part.

I have a suspicion that if geometry were attacked from this point of view in girls' schools it would be found to be more worth doing. In fact when some suggestions which I made along these lines were acted on there was a very marked increase in interest and in the quality of the work done, and actually one of the best courses in what seemed to me to be real geometry that I have ever seen was given in a dressmaking class at Shoreditch Technical Institute. And I cannot think that boys are so entirely different from girls that, *mutatis mutandis*, they would not be the better for a little more attention being given in boys' schools to appreciation of space relations, which includes, of course, relations in a plane.

There is another point that seems to me to be of importance and which perhaps may touch you even more closely. It is a fact that the whole of logical geometry is based on intuitions, but we call them postulates. I have a suspicion that it would be well if, at first, we assumed as evident a little more than we do. Some improvement has been made ; we do not now begin geometry with an attempt to prove what any small boy would assume as evident, that the angles at the base of an isosceles triangle are equal, an attempt which did more to put boys of my day off geometry than anything else ; it was so obviously a silly business.

A very sound principle in the teaching of geometry as a logical scheme is to prove something only when the boy begins to doubt it. As long as it is obvious leave proofs alone. It is just here that the area properties of rectangles and triangles are so important ; not only is the whole of Euclid's first book built round these, either as preparation or deduction, not only was this the reason for geometry at all (arising as it did from the necessity for the measurement of areas in ancient Egypt) but the facts are *not* seen by intuition, the boy sees the sense in proving them, and the reasoning involved is simple.

Just what is obvious and what is not obvious depends on the strength of one's mathematical ability. The weaker the ability the more obvious are the facts and the greater the number of postulates, the better the mathematician the fewer are the postulates. My late chief, Sir Percy Nunn, once went about quite miserable for months because he could not prove that the line joining two points on either

side of a plane passed through the plane. He was so good a mathematician that he wished to prove it, and I think did ; I am so much poorer a mathematician that I took it for granted and did not worry about the proof.

But having, at last, got on to something that you call geometry, let me make another attempt to say something that you will think worth while.

Angles. One of the difficulties in the introduction of angles in a mathematical course is that usually two things have to be done at once :

(1) pupils are asked to work with angles, and
(2) they are supposed to understand what an angle is.

It is not altogether a bad plan in some cases to learn by, and while, doing, but in this case it seems to lead to permanent misunderstanding rather than to understanding. Possibly a little care in the geography course could ensure that the pupils are clear as to the meaning of an angle *before* accurate work with angles is undertaken in the geometry course.

One of the first geographical ideas to be introduced is that of direction, and of this the young people soon get fairly accurate conceptions. They know that north is exactly opposite south and that east and west are exactly half a turn between the two. A boy points to the sun at midday (G.M.T.)—to south as he thinks. He points to north in exactly the opposite direction. He turns half round and points to east and west and places that lie in those directions. It does not greatly matter whether or not he pointed at first exactly to south ; he knows how much he has to turn to point to the other directions. Turn! He gets at once the real fundamental idea of what an angle *is*, the amount of " turn ", the amount *he* turned. Boys and girls soon get a very good working knowledge of what a right angle is without necessarily knowing its name or being hampered by ideas of the lengths of the lines bounding the angle. They can, and do, soon know how much to turn to face the intermediate directions such as north-east.

After a year or so of such incidental work, more formal study may be begun by the construction of a compass card. Observations of the altitude of the midday sun may also be made by a simple form of angle-meter. The sides of a vertical wooden quadrant are 18 inches long. An iron rod is fixed horizontally on this, and another, free to turn round one end of the fixed one, has two thin brass pins $1\frac{1}{2}$ inches long projecting from it a few inches apart, perpendicular to the plane of the quadrant. It is obvious that, when the sun is in the plane of the vertical quadrant and the movable rod is turned from a horizontal plane till the shadows of the two pins on it coincide, it is pointing to the sun and has *turned* through the angle which is the altitude of the sun. In order to record this altitude it is necessary to divide the right angle into degrees and at that stage the purely mathematical work with angles can be introduced.

This, it seems to me, is an example of correlation in which work which is primarily mathematical is best done in the geography course. What I have suggested can of course be done in the mathematical classroom, but in that case there is a tendency to hurry which is fatal, and there is not time for the fundamental idea of "turn" to sink in and be taken for granted; it is only when a thing is taken for granted that it is really known. "Practical" geometry seems to me to do more harm than good in this case, just because an angle is almost always associated with its boundary lines and the amount of turn is rarely if ever mentioned. In the geography classroom, exercises with directions can, and should, be done again and again till they are known; the word "turn" is used again and again, the idea soaks in, and soaks in all the more effectively because the kinesthetic sense is exercised. In fact an angle *is* measured only by this kinesthetic sense. I believe that half, or more, of you think, or, what is more to the point, *feel*, that these bounding arms of an angle are in some way necessary and that if they are not there the angle does not exist. This is quite wrong.

Then, to digress somewhat from a logical presentment of what I have to say, there are *graphs* and *growth curves*. I think that this is another case where the geographer might be asked to introduce the idea. I know that Sir Percy Nunn rather inclined to the use of biological material, *e.g.* a growing bean, but personally, and I think I have reason on my side, I think that the best introduction to graphs is to plot the readings of a thermometer every hour or half hour during a school day to show the rise and fall of temperature (and incidentally, though this has nothing to do with graphs, to get the boys to begin to associate a particular temperature reading with a particular feeling, so that they have a standard by which to measure the temperatures that are felt elsewhere in the world).

The method of plotting is important. I would not—and did not—plot merely a dot when the thermometer was read at, say, 9 a.m. Each boy drew a picture of the thermometer, graduations and all, *and* the mercury. At 9.30 or 10 it was drawn again, and so on through the day at regular intervals of time and paper, so that at the end of the day there was a row of thermometers showing the heights of the mercury at the different times of reading. The general run of the temperature was noticed. Then came the questions, if the readings were taken every half hour: "What was the temperature at 9.15?" "Do you really know?" "Did anyone see it?" "Can you be sure if you did not see it?" etc., etc., and the value and limitations of such graphical work is emphasised.

Naturally observations of the same kind will be made again and again on other days to drive the thing home. Then, it seems to me, is the time for the mathematics master to step in and get the boys to find out that though they plot only a few points of a mathematical graph they can be *quite* sure of what the intermediate readings would be—one up for mathematics.

Perhaps I might point out that the plotting of temperatures at different times of one day is free from the complications of plotting temperatures at 9 a.m. each day, or the average for a day, or the mean for a month, the concrete meaning of which is not very clear even to adult minds. Readings of temperatures *through* a day gives a curve which is obviously continuous. As to averages, I have some sympathy with the boy who said that an average is what a hen laid an egg on. (He had been told that " a hen laid an egg on an average two days out of three.")

Much later the plotting of graphs of rainfall over a period may possibly help in the first stages of the calculus if the rainfall figures for the months of a, or the, year are compared with those for the weeks or days of a, or the, year. But I am not so happy about this and merely suggest it for your consideration.

Now, after that digression, let me go back to real earth measurement, surveying. From the geographical point of view this mainly means maps. The difficulty in map work is in map reading, *i.e.* in visualising what the map tells you, and it *is* a difficult job. But I will not go into that. I will only say that it helps a little if the pupils have taken a hand in surveying, surveying in both senses, in the generally accepted sense of making maps, and in the wider one of going a " look see ", perhaps comparing what is seen with the map that represents it. It is the technical sense in which mathematicians are, or ought to be, interested. Indeed I am so strongly of opinion that they ought to be interested that I think that it is they and not the geographer who ought to do the surveying, and in fact when I was teaching in one school I got one of my mathematical colleagues to do it.

That being so, it may seem impertinent of me to suggest *what* should be done and how it should be done, but after all I was once senior mathematical master in a school of 600 pupils and should know a little about it ; so let me say that I began surveying in the playground and that I did not use either a theodolite or a plane table, nor did I measure offsets from a central line. What I did do was to divide the boys into groups of four or five and give each group a foot-rule and a piece of string ; the latter was a little more than five yards in length which the boys themselves marked off into yards with a blob of ink at each yard.

With this equipment each group measured a base line and the lengths of the sides of the necessary triangles. The lengths were measured as multiples of five yards and what was over was measured in yards (always of course less than five), and the odd bit with the foot-rule in feet (always less than three) and inches. With a ten yard string there is a probability that a mistake will be made with the odd yards. One boy to each end of the string, the clerk and the foreman make four. Either the foreman can mark the end of each five yards or a fifth boy can be employed at that job.

I may add that on one occasion I tested the accuracy of the map made in this way against one made by a gang with a theodolite and

another made by a gang with a plane table, and there was no comparison between them. The string and foot-rule gave much more accurate results; in fact the maps made in this way were very accurate indeed. In addition this method gives a far greater number of boys a hand in the work.

But even more important, this seems to me to put the angle business in triangulation, a great part of geometry, on a sensible footing; you measure distances when you can, and use angles when for any reason it is impossible to measure distances correctly.

The ordnance map of the school district did not show a number of roads as these had been constructed since the map was made: for a reason which will appear later it was necessary to have these roads on the map, but it was impossible to measure the necessary distances as houses had also been built; boys from a higher class were, however, sent out to run a traverse round the different blocks with a theodolite and chain. This, of course, means fixing triangles by means of two sides and an included angle.

All that is two-dimensional map construction and, as I have suggested, I like three dimensions. Thus the next thing is to insert contour lines. The map made is of a larger area than a playground, and a water level and a chain come into the picture. The level consists of:

(1) A tripod, to which is attached by a bolt and fly nut

(2) a piece of wood, *BC*, 30 in. by 3 in. by 1 in., with two circular holes, one at each end, to receive

(3) the essential part of the instrument, two glass tubes, *DE*, 1 in. in diameter, connected by a rubber tube. The connections at *D* are rubber stoppers, bored to receive small glass tubes, on which the rubber tube is slipped.

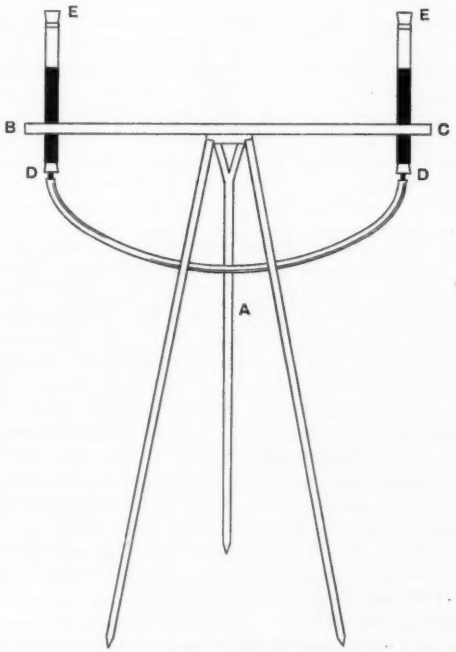
EE are two rubber stoppers, bored to receive small tubes bent at right angles; these allow of free communication with the air, and yet if turned in the direction opposite to that from which the wind is blowing, prevent sudden gusts from affecting the level of liquid in the tubes. If it is desired to make sure that the wind shall have no effect these tubes can be connected by a rubber one.

Water, coloured with ink, is poured into *DE* and finds its level in both tubes. If one stands some distance from the level, so that one of the tubes is behind the other, and then lowers the eye till the surface of the farther liquid is in line with that of the nearer, then the eye is at the same height as either surface. *So also is any point seen behind the farther surface.* By using this fact, levels at different points may be found. The advantages of the instrument are that it is simple to work and its working is easily understood.

With it we can find all points at one height, *i.e.* a contour line. I have done this by taking a class out on to Parliament Hill and standing the boys a yard apart with their feet at the same height; the level is placed at one end of the line and sighted on to each boy's feet in turn so that they are level with the surfaces of the water in

the two tubes, as determined by an eye a yard or so away from the nearer one.

This obviously does not make a map. A map can, however, be made if points on a contour line in the field can be identified in some way with the corresponding points on a map. This was done by fixing the points by observations from the two ends of a measured base line recorded on a plane table at each end, the angles between the direction of the point and the base line being measured. In fact each triangle was fixed because two angles and the side between them was known. (Incidentally this is the third method of fixing a triangle.) The mark to be sighted was one on a pole instead of a boy's feet.

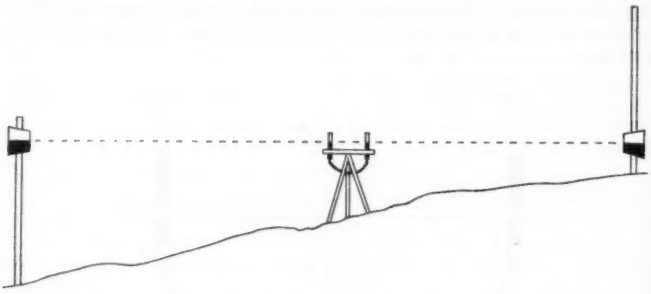


In practice, however, partly because the plane table work was troublesome, we did not insert contour lines on the map in this way. We found it simpler to run lines of levels along courses that could be identified on the map. This meant finding, not a series of points all at the same height, but a series in which each point was higher or lower than the last by a fixed amount, in this case 5 feet.

It was also convenient to sight on to a frame which slid up and down on a pole marked in feet and inches. The frame was a foot

square with the top half white and the bottom half black, the division being, of course, horizontal ; this division made an excellent "line" on which to sight ; in favourable conditions it could be seen half a mile away. The frame could be moved up and down, or the pole could be moved about till the sighting line was exactly on a level with the two surfaces.

The principle involved is simple. Suppose a point is known to be 150 ft. above sea-level and we are going uphill. The pole is placed



at this point. The level is placed some distance away on the "up" side of the pole and sighted on to the division on the frame, which is moved up and down on the pole till it is in line with the levels in the tubes ; the reading is, say, 6 ft. 7 in. The frame is now lowered 5 ft. to 1 ft. 7 in. and clamped ; then in turn it is transferred to the "up" side of the level and moved backwards and forwards till the "line" on the frame is found to be at the same height as the water in the tubes. This point is 5 ft. higher than the previous one and is therefore 155 ft. above sea-level. A little piece of geometry makes this evident. The distance between the points can be chained. If one is on a definite line such as a footpath and the position of the 150 ft. point can be identified on the map, the position of the 155 ft. point can also be identified on the map and plotted. By starting from points such as bench marks marked on the ground and on the corresponding maps, lines of levels may be constructed in which the heights of points above sea-level are known.

These lines may be either

(*a*) paths, as I have already suggested ;

(*b*) lines in open country, of which the two ends can be fixed and a constant direction from the one to the other can be kept ;

(*c*) streets ; here is the advantage of having streets inserted on the map.

The streets supply very convenient networks, but there is a snag, in fact two snags. One is that it is a little difficult to make long shots with a good number of people on the pavements ; the other is that these people tend to gather in great numbers to see what is

happening, and a policeman may also be drawn to the scene so that *he* also can see what is happening. It is then advisable to efface oneself and the boys, and retire into obscurity or some other part of the district for a while till the trouble blows over.

Then there are latitude and longitude networks and projections generally, which are obviously of interest to the geographer but which in themselves are so mathematical that they equally obviously should be " done " in the mathematics course.

The trouble is that most of the ordinary geometrical projections of latitude and longitude networks, while useful and interesting mathematically, are of comparatively little use for map work, and the networks useful to geographers are too complicated mathematically to be of use in elementary mathematical work.

It is a pity that there is this difficulty, for otherwise it would be well if the mathematics staff could see to the construction of latitude and longitude networks for use in geographical work. It is just possible that the geography master would welcome a simple conical projection and perhaps even more warmly a sinusoidal. Anyhow do not think that the latter is impossible in a middle form. I have seen a standard VII in a senior school, not the master but the boys, construct one on the blackboard in a three-quarters of an hour lesson ; then they each made one for themselves. It is only fair to say that the " master " was Sir Percy Nunn so that perhaps some allowance should be made, but anyhow your fourth or fifth forms should be able to do it easily.

Even if few networks are constructed, some light may be thrown on the problems involved by an ingenious device also used by Sir Percy for observing and recording the path of the sun in the heavens during a day. " A number of thin rods (*e.g.* long knitting needles) are mounted perpendicularly at equal intervals along the circumference of a circle marked on a drawing board. Each rod carries a small paper or cardboard slider. The board is fixed horizontally in sunshine. As from time to time during the day the shadow of one of the rods falls across the centre of the circle the slider is so adjusted that its shadow covers the centre." When at the end of the day the sliders are all in place we have a graphic record of the path of the sun across the heavens. Records for other days may be constructed in a similar way. Then if permanency is desired it is necessary to enter the heights registered on a sheet of paper whose length is equal to the circumference of the circle and on which the needles are represented by lines vertical to it. Here in fact we have a latitude and longitude network. If the board is tilted to be parallel to the plane of the equator you have some very pretty results and an introduction to sundials.

It is possible also for upper classes to give some reality to the ideas of latitude and longitude by finding, fairly accurately, the latitude and longitude of the school. This is badly wanted in the geographical work and the mathematicians can help materially. Actually, as geography master, I used to get my top boys to do the

work, but not all geography masters are mathematicians and it was not really my job.

There are two methods of finding the latitude. One of these is quite simple but requires observations of the altitude of the mid-day sun throughout the year at such times as will give a continuous curve, with time and altitude as "x" and "y". The actual observations are those suggested as coming into elementary work with angles. There the main idea is to get the boys to "*observe*" that the sun does in fact change its midday altitude during the year and to correlate it with change of temperature. The plotting of the results to obtain the latitude must be left till a few years later. When this is done it is easy to show that the horizontal line which divides the curve into two equal parts gives the latitude. It is a simple little piece of circle work.

The other method is to plot the altitude of the sun at different times through one day, the *times* being very carefully noted. This gives both latitude and longitude. The latitude is easily found from the maximum altitude when the declination of the sun is allowed for, and again supplies a simple little piece of work with circles. Finding the longitude is a little more tricky; it is comparatively easy to find the greatest altitude of the sun but difficult to tell exactly *when* it is greatest. In fact one does not attempt to find it directly. Instead, one attempts to find when it is midday at the place of observation and compare it with the time of midday at Greenwich. The difference, of course, gives the longitude. Midday at Greenwich, the middle of the day there, halfway between sunrise and sunset—not, by the way, usually at 12 o'clock—can be found from Whitaker's tables. The times of sunrise and sunset at the place of observation are not so easy to find directly, but mid-day may be found by noting the times before noon and after noon when the sun is at the same altitude. These and the time midway between them may easily be read from the graph of observations, and if a few of these mid times, *not* near midday, are noted and averaged, quite reasonably accurate results are given. Instead of finding midday at Greenwich the correction for the equation of time may be made to G.M.T.

Another interesting but very simple piece of work gives the size of the earth. If two schools, one some distance north of the other, observe the midday altitude of the sun through a year and exchange graphs, it will, of course, be found that there is a constant difference, and if the north-south distance between them is known, as it can be found from a map, a very simple piece of work with a circle gives the size of the earth. Single observations on the same day will naturally gives the same result; this, in our climate, is a little chancy, but if a few observations at each school, not necessarily on the same days, are made in the course of a fortnight or month curves may be drawn and the constant difference found. It was in fact because Eratosthenes noted the midday difference in the altitude of the sun at Alexandria and at Aswan, on the tropic where

the sun is overhead at midday in midsummer, that he found the size of the earth, and knew it a great deal more accurately than did Columbus.

And do not think that this requires elaborate apparatus. Eratosthenes had only a vertical stick standing in a hemispherical bowl, but he knew very accurately the distance between Aswan and Alexandria, as it had all been very carefully measured. I in Scotland and a mathematical colleague in the Isle of Wight found the size of the earth very accurately one summer in this way, and all that I had was a piece of wood picked up on the beach, a piece of paper, a pin, a stout thread and a key. The paper was fastened to the wood, the pin was at one corner, and the key was hung from it by the thread. A line was drawn on the paper through the pin. The key and string formed a plumb bob which secured that the line through the pin should be vertical when an observation was made. The wood was held in sunshine so that the pin cast a long shadow across it, and a dot was made where it fell a good distance from the pin, the date being put alongside it. Three or four midday observations were made and when the points were joined to the pin later, the angles were measured with a protractor.

Needless to say all this correlation in earth measurement makes GE-ometry a much more vital affair than the usual demonstrations.

So much for all that. Let me conclude with the barest outline of a piece of correlation which takes in physics, geography and mathematics : it concerns the study of rainfall.

First, a piece of physics which usually does not come in the physics course because it is too simple (I wonder!) A saucer of water is left and seen to evaporate ; so, there is water in the air even if we do not see it.

Then there is a piece of pure geography, the fact that there is a variation in the amount of rainfall at different places.

Thirdly, there is the measurement of rainfall, purely mathematics, involving, as it does, areas of circles and volumes of cylinders.

Fourth, the causes of rainfall, physics.

And lastly, the distribution of rainfall and the causes acting in different places, geography.

And there I stop. Perhaps I have not stuck to my terms of reference as closely as I might have done, but if you have at all benefited from what I have said I shall feel that I have done well ; also, if I have trailed my coat a little, even though as far as I know there is no Irish blood in my veins, it is just to make you think, an excellent exercise. J. F.

1320. "The editor [of 'Straw in the Hair, an anthology of nonsensical and surrealist verse', edited by D. K. Roberts] . . . defends nonsense as a product of the imagination. 'Nonsense', he writes, 'is the name given by knowledge to imagination's account of its travels in regions uncharted by practical experience.' This is a pretty big claim. It might also be a definition, for example, of mathematics."—A. G. Macdonell, in *The Observer*, December 25, 1938. [Per Mr. E. S. Pondiczery.]

MATHEMATICAL NOTES.

1460. *Formulae for the solution of triangles.*

The writer feels very strongly that when these formulae are first introduced to pupils it should be as far as possible by geometrical methods, rather than by somewhat complicated algebraic manipulations in which the student is apt to lose sight of the fact that trigonometrical formulae are expressions of geometrical relations.

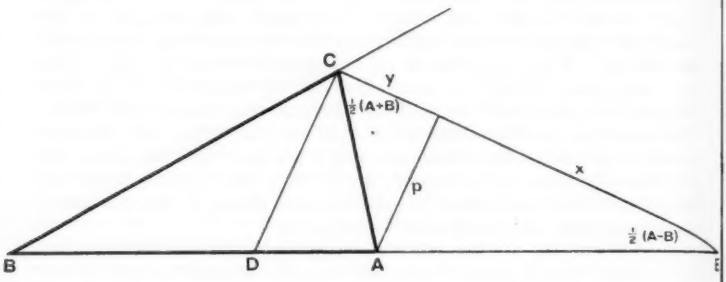


FIG. 1.

The lengths of the segments of the base by the bisectors of the internal and external angles at C are found to be

$$BD = ac/(a+b),\ DA = bc/(a+b),\ AE = bc/(a-b),\ BE = ac/(a-b).$$

Hence
$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{p}{x} \Big/ \frac{p}{y} = \frac{y}{x} = \frac{DA}{AE}$$
$$= \frac{bc}{a+b} \Big/ \frac{bc}{a-b}$$
$$= \frac{a-b}{a+b}.$$

In the triangle BCE
$$\frac{a}{ac/(a-b)} = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$$

or
$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

In the triangle BCD
$$\frac{a}{ac/(a+b)} = \frac{\sin CDB}{\sin\frac{1}{2}C}$$
$$= \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$$

or
$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

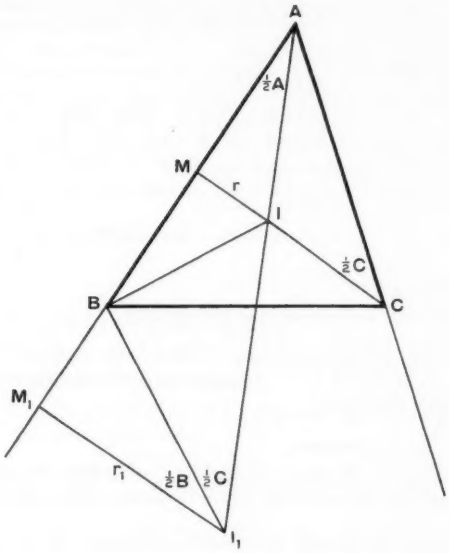


FIG. 2.

The point I being the in-centre and I_1 the ex-centre opposite A, the lengths of the segments on AB are

$$AM=s-a,\quad MB=s-b,\quad BM_1=s-c,\quad AM_1=s.$$

$$\tan\tfrac{1}{2}B=r/(s-b)=(s-c)/r_1$$

and so
$$rr_1=(s-b)(s-c).$$

The triangles ABI_1, AIC are similar, and so

$$c:AI=AI_1:b$$

or
$$AI\,.\,AI_1=bc.$$

Thus

$$\tan\tfrac{1}{2}A=\frac{r}{s-a}=\frac{r_1}{s}$$

$$=\sqrt{\left\{\frac{rr_1}{s(s-a)}\right\}}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}};$$

$$\sin \tfrac{1}{2}A = \frac{r}{AI} = \frac{r_1}{AI_1}$$
$$= \sqrt{\left\{\frac{rr_1}{AI \,.\, AI_1}\right\}} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}};$$
$$\cos \tfrac{1}{2}A = \frac{s-a}{AI} = \frac{s}{AI_1}$$
$$= \sqrt{\left\{\frac{s(s-a)}{AI \,.\, AI_1}\right\}} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}};$$
$$\Delta = rs = r_1(s-a)$$
$$= \sqrt{\{s(s-a)rr_1\}}$$
$$= \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

A. HINCKLEY.

1461. *Problem.*

Prove that the n-dimensional polyhedron

$$|x_1| + |x_2| + \ldots + |x_n| + |x_1 + x_2 + \ldots + x_n| = 2$$

has $(2^{n+1} - 2)$ faces of $n-1$ dimensions and contains an n-dimensional volume $(2n!)/(n!)^3$. L. J. M.

1462. *Products of vectors.*

Disregarding (as Kipling might have said) the inventions of the Irish Astronomer whose other name is Rowan, let a plain statement suffice.

Two vectors of moduli r, r' and amplitudes θ, θ' have THREE products.

I. A vector of modulus rr' and amplitude $(\theta + \theta')$. We learn about this in any large book about pure mathematics or higher trigonometry or higher algebra.

II. A scalar; the dot-product; the number $rr' \cos(\theta - \theta')$.

III. A vector; the cross-product; its modulus is $rr' \sin(\theta - \theta')$. It is perpendicular to the plane of the two vectors in a direction which depends on the order in which they are named.

These two *do not depend on the orientation of the axes,* which is perhaps why they are more valuable in problems of physics than No. I. We learn about these two in any book on vector analysis.

But it is expensive in books, even "disregarding the inventions . . . ". C. O. TUCKEY.

1463. *Note on γ.*

I suppose that the most exciting result a boy meets in an introductory course in limits and convergence is the fact that the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots$$

is divergent. It follows from the usual method of showing the

divergence that, if m is a positive integer, the sum of 2^m terms exceeds $1+\frac{1}{2}m$, and it can be shown as easily that this sum is less than $1+m$. Thus the sum of 2^{18} terms of the series is greater than 10 and less than 19; so the series must diverge very slowly. Usually some curious voice will ask just how many terms must be taken before their sum exceeds 10. The facts that an exact answer, 12,367, can be given which is large enough to preclude the likelihood of its being reached by term-after-term addition and that yet there exists no exact formula for the sum of n terms of the series, leave matters in a state of mystery, which should certainly be cleared up as soon as the logarithmic series have been dealt with. The process of clearing up the mystery is interesting and makes an excellent revision lesson.

It involves (*a*) showing that

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}-\log n$$

tends to a limit γ as n tends to infinity;

(*b*) the calculation of γ with a determination of how nearly the limit is approached as n increases.

(*a*) By considering rectangles and trapezia and the area under the curve $y=1/x$ we can show that

$$\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}<\int_1^n\frac{dx}{x}<\tfrac{1}{2}(1+\tfrac{1}{2})+\tfrac{1}{2}(\tfrac{1}{2}+\tfrac{1}{3})+\ldots+\tfrac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n}\right),$$

that is,
$$s_n-1<\log n<s_n-\tfrac{1}{2}-\frac{1}{2n};$$

hence
$$1>s_n-\log n>\tfrac{1}{2}.$$

Also
$$\{s_{n-1}-\log(n-1)\}-\{s_n-\log n\}=s_{n-1}-s_n-\log(n-1)/n$$
$$=-\frac{1}{n}+\frac{1}{n}+\frac{1}{2n^2}+\ldots$$
$$>0.$$

This is otherwise obvious, as the left-hand side is the area bounded by $x=n-1$, $y=1/n$ and the curve $y=1/x$. Thus $\{s_n-\log n\}$ decreases steadily as n increases. Since it exceeds $\frac{1}{2}$ for all values of n, it must tend to a limit $\gamma\geqslant\frac{1}{2}$ as n tends to infinity.

(*b*) Usually an approximate value of γ is stated in school algebras without any indication of how it can be calculated. The best method, I suppose, is one that uses the asymptotic expansion

$$\gamma=s_n-\log n-\frac{1}{2n}+\sum_1^k\frac{(-)^{r-1}B_r}{2rn^{2r}}+(-)^kR_k,$$

where $R_{k-1}<B_k/2kn^{2k}$, but this is of course beyond the schoolboy. For him the following method is suggested, either in stages as an exercise or as a revision lesson.

Since $$\log(n+a)-\log n\to 0 \text{ as } n\to\infty,$$
$$s_n-\log(n+a)\to\gamma \text{ as } n\to\infty,$$

and we may write $s_n=\log(n+a)+\gamma+\epsilon_n$, where $\epsilon_n\to 0$ as $n\to\infty$, and a is an arbitrary constant which will be fixed later.

Thus $$s_{n-1}=\log(n-1+a)+\gamma+\epsilon_{n-1},$$

and, subtracting,

$$1/n=\log(n+a)/(n-1+a)+\epsilon_n-\epsilon_{n-1};$$

by inspection when $a=\frac{1}{2}$ terms in $1/n$ and $1/n^2$ disappear, for we then have

$$1/n=\log(2n+1)/(2n-1)+\epsilon_n-\epsilon_{n-1}.$$

Thus $$\epsilon_{n-1}-\epsilon_n=2\left\{\frac{1}{3}\cdot\left(\frac{1}{2n}\right)^3+\frac{1}{5}\left(\frac{1}{2n}\right)^5+\dots\right\};$$

$$\frac{1}{12n^3}<\epsilon_{n-1}-\epsilon_n<\frac{1}{12n^3}\cdot\frac{1}{1-(1/n^2)}=\frac{1}{12(n-1)n(n+1)};$$

$$\sum_n^m\frac{1}{12r^3}<\epsilon_{n-1}-\epsilon_m<\sum_n^m\frac{1}{12(r-1)r(r+1)};$$

$$\sum_n^\infty\frac{1}{12r^3}<\epsilon_{n-1}<\frac{1}{24(n-1)n}, \quad \text{since } \epsilon_m\to 0 \text{ as } m\to\infty.$$

Although a low approximation to $\sum_n^\infty 1/12r^3$ is evidently $\int_n^\infty dx/12x^3$, we can fix ϵ_{n-1} more precisely by summing the term-by-term differences of the bounding series thus:

$$\frac{1}{(r-1)r(r+1)}-\frac{1}{r^3}=\frac{1}{r}\left\{\frac{1}{r^2-1}-\frac{1}{r^2}\right\}$$
$$=\frac{1}{r^3(r^2-1)}$$
$$<\frac{1}{(r-2)(r-1)r(r+1)(r+2)};$$

hence $$\frac{1}{2(n-1)n}-\sum_n^\infty\frac{1}{r^3}<\frac{1}{4(n-2)(n-1)n(n+1)},$$

and so ϵ_{n-1} is less than $1/24n(n-1)$
by less than $1/48(n-2)(n-1)n(n+1)$.

By choosing a numerical value of n, say 12,

$$s_{12}=\log 12\tfrac{1}{2}+\gamma+\epsilon_{12}.$$

Now $$s_{12}=3{\cdot}103210678\dots,$$
$$\log 12\tfrac{1}{2}=2\log 10-3\log 2$$
$$=2{\cdot}525728654\dots,$$

and so $$\gamma+\epsilon_{12}={\cdot}577482024,$$

where ϵ_{12} is less than ·0002671 by less than ·0000009, and so

$$\cdot 5772149 < \gamma < \cdot 5772159.$$

[Actually $\gamma = \cdot 57721566 \ldots$.]

So we have calculated γ to within an error of 5 in the 7th decimal place and have found that

$$s_n = \log (n + \tfrac{1}{2}) + \gamma + 1/24n(n+1) - \delta,$$

where $$\delta^{-1} > 48(n-1)(n)(n+1)(n+2).$$

To answer the question " what is the least number of terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ which must be taken for their sum to exceed 10? " we proceed thus :

(i) we show that $$s_{10000} < 10,$$

and when $n > 10000$, then

$$\{24n(n+1)\}^{-1} - \delta < 10^{-9},$$

which may be neglected in finding a first approximation to a solution of the equation in n,

$$10 = \log (n + \tfrac{1}{2}) + \gamma + \{24n(n+1)\}^{-1} - \delta ;$$

(ii) we then solve

$$10 = \log (n + \tfrac{1}{2}) + \gamma,$$

which gives $$n + \tfrac{1}{2} = \exp (9 \cdot 42278 \ldots).$$

Seven-figure logarithms give n quickly, but it may be calculated without them by noticing that $8 \log 10 = 18 \cdot 42068$,

i.e. $$10^8/e^9 = \exp (9 \cdot 42068 \ldots),$$

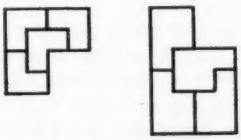
calculating e^9 (*via* e^2, e^4, e^8) and exp (·00210) from the exponential expansion. Now $(n + \frac{1}{2})$ comes to just under 12367, from which it follows that the sum of 12366 terms is less than 10 and the sum of 12367 terms is greater than 10.

R. C. Lyness.

1464. *Uses of a geometric puzzle.*

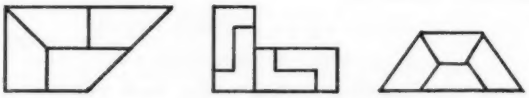
The form of puzzle of which examples are given here can be used to show the relation between linear dimensions and areas of similar figures.

It is well known that four squares, triangles or parallelograms may be fitted together to make one figure similar to but larger than the original. Puzzle makers and others probably also know of the Figures 1 and 2.



Figs. 1, 2.

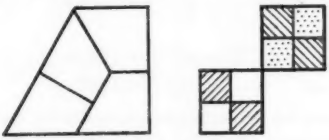
Figs. 3-7 give some others less well known; that in Fig. 7 always seems to me to be mean. Figs. 3 and 6 are Figs. 1 and 5 bisected. Fig. 5 itself is half a regular hexagon.



FIGS. 3, 4, 5.

A class can be shown a few of these and then asked to find others.

The problem may, if desired, be extended to solid figures. I have not yet found a really pleasing solid corresponding to either Fig. 2 or Fig. 4; a prism with either of these as base will, of course, satisfy the condition required, but this seems more like using thick cardboard in place of paper than finding really new solutions.



FIGS. 6, 7.

There is also the problem of fitting together nine plane figures; thus, for example, Figs. 1 and 5 give the new Figs. 8 and 9, but Figs. 2, 4 and 7 do not seem to extend thus.

FIGS. 8, 9.

Apart from extensions to 16 or 25 plane figures or 27 solids, here are a few questions for those who like such things:

(i) Are there more figures like Figs. 1-7, or can it be proved that there cannot be any more?

(ii) Can conditions for 4, 9, 25 ... figures to fit together be found, as with Figs. 1 and 8, or Figs. 5 and 9, and can conditions for a figure to fit in isolated cases, as in Fig. 2, be found?

(iii) Can a figure be found such that 9 of them fit together into a larger similar figure, but not 4 of them? If not, can the negative be proved?

(iv) Questions for solids, similar to (i)-(iii).

(v) What would the corresponding four-dimensional problem be?

C. DUDLEY LANGFORD.

1465. *Lodge's theorem on the motion of the centre of gravity.*

In Note 753 (*Gazette*, No. 175) the late Prof. Alfred Lodge puts in a plea for the following rule: "If $1/n$th part of a mass has its centre of gravity moved through a distance d, the centre of gravity of the whole mass is thereby shifted a parallel distance d/n."

We have several useful working rules for moments of inertia; the above theorem, for centre of gravity, which might be appropriately called Lodge's rule, can hardly be considered either a strain on the memory or a low-down, mark-scoring short cut.

The proof is easy enough for a fifth form. With the usual notation

$$\bar{x} = \Sigma m_r x_r / \Sigma m_r, \quad \bar{y} = \Sigma m_r y_r / \Sigma m_r.$$

If m_p is moved from (x_p, y_p) to $(x_p + h, y_p + k)$ then the new centre of gravity $(\bar{x}', \bar{y}')$ is given by

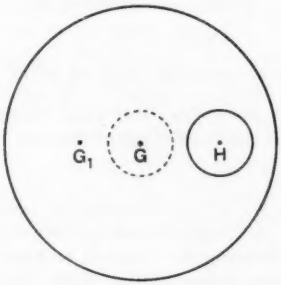
$$\bar{x}' = (\Sigma m_r x_r + m_p h)/\Sigma m_r = \bar{x} + m_p h/\Sigma m_r,$$

$$\bar{y}' = (\Sigma m_r y_r + m_p k)/\Sigma m_r = \bar{y} + m_p k/\Sigma m_r.$$

So if $m_p = \Sigma m_r/n$, the new centre of gravity is $(\bar{x} + h/n, \bar{y} + k/n)$.

As an example; if a 16-st. passenger moves from one side of the rear seat of a 1-ton car to the other, the centre of gravity of the whole is shifted about 2 inches. The theorem is useful in tracing the motion of the centre of gravity of a body, part of which is folded about a line, as when the lid of a box is opened. It also gives an easy way of dealing with the centre of gravity of a body from which a part has been removed, or with an almost symmetrical body.

Suppose the body to have mass M, centre of gravity G; the portion removed to have mass m, centre of gravity H. If we cut out from the remainder a piece whose centre of gravity is G and mass m, and move it into the "hole", we shall restore the centre



of gravity of the remainder to G. But as we shall have moved $m/(M-m)$ of the mass a distance GH, the centre of gravity of the remainder must have been at G_1, in HG produced, where

$$GG_1 = m/(M-m) \,.\, GH.$$

I have found that boys take readily to the use of this rule.

A. P. ROLLETT.

1466. *Odds and evens.*

The use of $(-)^n$ to produce the sequence $-, +; -, +; \ldots$, and, more rarely, of $(-)^{\frac{1}{2}n(n+1)}$ to produce $-, -, +, +; \ldots$, suggests an investigation into the sequence of odd and even integers generated by algebraical formulae. For instance, if

$${}_nH_r \equiv (n+r-1)!/(n-1)!\,r!,$$

then $(-)^{nH_r}$ gives the following repeated patterns in signs:

$r=1$, $-, +; -, +; \ldots.$
$r=2$, $-, -, +, +; \ldots.$
$r=3$, $-, +, +, +; \ldots.$
$r=4$, $-, -, -, -, +, +, +, +; \ldots.$
$r=5$, $-, +, -, +, +, +, +, +; \ldots.$
$r=6$, $-, -, +, +, +, +, +, +; \ldots.$
$r=7$, $-, +, +, +, +, +, +, +; \ldots.$
$r=8$, $-, -, -, -, -, -, -, -, +, +, +, +, +, +, +, +; \ldots.$
$r=9$, $-, +, -, +, -, +, -, +, +, +, +, +, +, +, +, +; \ldots,$

and so on. Perhaps others will produce a general method, or particular cases not occurring above, for instance, the sequence $-, -, -, +, +, +$.

A. P. ROLLETT.

1467. *Braking efficiency in motor cars.*

The Brake-Testing Service for motor cars gives the vehicle a certificate of 100% efficiency if it can be stopped in 30 feet from a speed of 30 miles per hour. This will occur when the coefficient of friction is almost unity. For, from $Fs = \mu Ws = Wv^2/2g$,

$$\mu = v^2/2gs = 44^2/64{\cdot}4 \,.\, 30$$
$$= \tfrac{1936}{1932} = 1{\cdot}00.$$

The braking experts consider that an efficiency of 70% is "excellent", while even 50% is "good". It appears, then, that $\mu = \frac{1}{2}$ is average for a modern car, and that "braking efficiency" is numerically equal to the coefficient of friction. A. P. ROLLETT.

1468. *Icosahedron.*

I am indebted to Mr. W. C. Fletcher for pointing out to me that the twelve vertices of a regular icosahedron all lie on the faces of a cube, each vertex dividing in "golden section" the line from the centre of a square face to the middle point of a side.

This fact gives the clue to the drawing of a regular icosahedron in "oblique parallel projection", that is, the conventional and fairly realistic method of drawing a cube, in which the front and back are shown as squares, and the lines at right angles to them are oblique and foreshortened.

The following coordinates, from which the cube and icosahedron can easily be drawn, are based on the approximations 89 : 144, 21 : 34 and 29 : 47 to the "extreme and mean" ratio $\sqrt{5}-1 : 2$.

		x	y
Corners of front of cube		0	0
		0	144
		144	144
		144	0
Corners of back of cube		47	34
		47	178
		191	178
		191	34
Corners of icosahedron.	Two on front	72	$27\frac{1}{2}$
		72	$116\frac{1}{2}$
	Two on back	119	$61\frac{1}{2}$
		119	$150\frac{1}{2}$
	Two on top	51	161
		140	161
	Two on floor	51	17
		140	17
	Two on right wall	153	$78\frac{1}{2}$
		182	$99\frac{1}{2}$
	Two on left wall	9	$78\frac{1}{2}$
		38	$99\frac{1}{2}$

By taking the means of the coordinates in each of the twenty sets of three adjacent vertices of the icosahedron, a regular dodecahedron can also be drawn. W. HOPE-JONES.

1469. *A suggestion in notation.*

If the expression $(AZBXCY)$ be evaluated in a similar way to a cross-ratio, the following is obtained: $(AZ \,.\, BX \,.\, CY)/(AY \,.\, CX \,.\, BZ)$.

Suppose, therefore, that we take X, Y, Z on the sides BC, CA, AB of a triangle ABC, we see that, putting:

$(AZBXCY)$ equal to minus one, gives Ceva's theorem;

$(AZBXCY)$ equal to plus one, gives Menelaus' theorem;

while if a conic cuts the sides BC, CA, AB of a triangle in X, Y, Z and P, Q, R, then Carnot's theorem is expressed by

$$(AZBXCY) \,.\, (ARBPCQ),$$

being equal to plus one.

If, therefore, $(AZBXCY)$ is in this case equal to minus one, so

also is $(ARBPCQ)$. Hence if AX, BY, CZ are concurrent, so also will AP, BQ, CR be concurrent.

Similarly, if $(AZBXCY)$ is equal to minus one, so also will $(ARBPCQ)$. Hence if X, Y, Z are collinear, so also are P, Q, R.

The notation considerably shortens the writing in solutions ; *e.g.* :

If P, Q, R are the harmonic conjugates of X, Y, Z in the sides BC, CA, AB of a triangle ABC, we immediately see that

$$(AZBXCY) \,.\, (ARBPCQ)$$

is equal to minus one. Hence if AX, BY, CZ are concurrent, then P, Q, R are collinear, and *vice versa*.

If a conic touches the sides of a triangle at X, Y, Z, then $(AZBXCY)$ squared is equal to one. Hence $(AZBXCY)$ is equal to either plus one or minus one.

Taking the minus sign, means that AX, BY, CZ are concurrent.

Taking the plus sign, the only possible interpretation is that the conic consists of two coincident lines, each being XYZ.

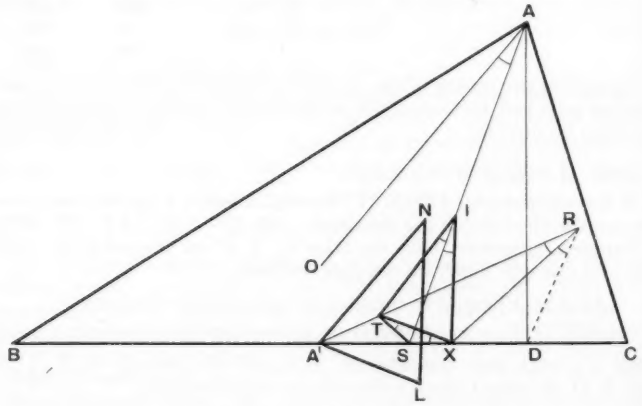
Lastly, if ABC is a triangle, and X, Y, Z points in BC, CA, AB, the expression $(AZBXCY)$ is projective, taking any vertex O.

H. N. HASKELL.

1470. *A proof of Feuerbach's theorem without use of inversion.*

(See Note 1234, *Gazette*, XXI, 1937, p. 154.)

Description of figure. Let the incircle touch BC at X. Let AI meet BC at S. Draw NL, a radius of the nine-points circle, parallel to IX. Draw ST to touch the incircle at T. Join I, T. Join A', T and produce it to meet the incircle at R. Join R, X and R, D.



Proof. This proceeds on the usual lines as far as proving that

$$A'S \,.\, A'D = A'X^2.$$

But $$A'X^2 = A'T \cdot A'R ;$$

hence T, R, D, S are concyclic.

Hence $$\angle TRD = \angle TSA' = \angle TIX = 2\angle TXS.$$

Also $$\angle TXS = \angle TRS.$$

Now, by symmetry, the tangent ST to the incircle is inclined to AC at an angle equal to $\angle ABC$, and hence ST is parallel to the tangent to the circumcircle at A, and also parallel to the tangent to the nine-points circle at A'.

Thus IT is parallel to NA'. It follows that the two triangles ITX and $NA'L$ are homothetic. Hence NI, AT, LX will meet in a point.

Now from above we see that RX bisects $\angle TRD$, that is, it bisects $\angle A'RD$. Also

$$\angle TRD = \angle TIX,$$

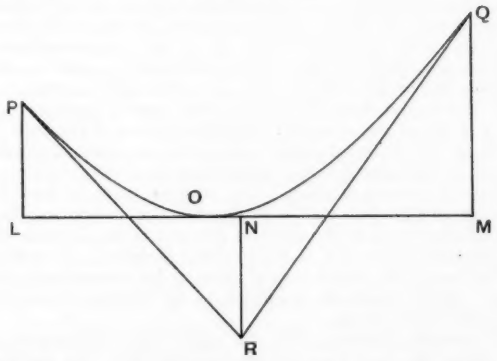
and so $$\angle A'RD = \angle A'NL.$$

Hence R lies on the nine-points circle. But the bisector of $\angle A'RD$ must pass through L, the middle point of the arc $A'D$. Hence $A'T$ and LX meet at R. Thus NI passes through R, which lies on both the incircle and the nine-points circle and is a homothetic centre of the two circles. This is only possible if the two circles touch at R.

H. N. HASKELL.

1471. *A geometrical problem.*

I should be very glad to have a solution of the following problem, which is about the only one of its kind for which a solution is not given in Eagle's *Geometry of Plane Curves*.



A parabolic arc is to pass from P to Q, and to touch LM at O. A geometrical construction is required for the tangents PR, QR. I am not interested in the focus of the curve, since I should draw it as an envelope; but I should be glad to have ON, the abscissa of R, obtained in terms of the (rectangular) coordinates of P and Q.

A. ROMNEY GREEN.

REVIEWS.

The Classical Groups, their Invariants and Representations. By HERMANN WEYL. Pp. xii, 302. 18s. 1939. (Princeton University Press, New Jersey ; Humphrey Milford)

The subjects discussed in this important book derive from several different theories, widely diverse in origin, which have each had to undergo a more or less extended period of independent development before they could be merged in a synthesis such as the present.

There are three principal ingredients involved. First, there is the theory of algebraic invariants and covariants, a " period " subject, now somewhat out of fashion. Secondly, there is the theory of semi-simple algebras, which has been for the last dozen years or so a particularly fertile field, and is now as a result of the fundamental discoveries of this period one of the most satisfying branches of modern algebra. Thirdly, forming something of a link between the former two, is representation theory.

In its original form, due to Frobenius, Burnside and Schur, representation theory had to do exclusively with groups of finite order. The classical groups referred to in the title, on the other hand, are all of them infinite groups. It is possible, however, by making use of what is here called the " commutator algebra " of a completely reducible matric algebra, to apply the representation theory of certain finite groups to the analysis of the representations of certain infinite groups. The most important instance of this is the analysis of tensors by means of symmetry operators. In this case, the finite groups concerned are the symmetric permutation groups, and they are used in order to split up the tensorial representations of the general linear groups into their irreducible components.

This analysis of tensors is given an important place in the present book. For the general linear group $GL(n)$, with its sub-group the special linear group $SL(n)$, consisting respectively of all non-singular or all unimodular linear transformations in n variables, are the classical groups which underlie the " classical " theory of algebraic forms. It is their representations and invariants which are studied in this theory, although it is true to add that, in the standard works on the subject, such as Gordan-Kerschensteiner or Grace and Young, this representational point of view scarcely emerges.

Indeed, it is one of the strongest recommendations of the book before us that it gives, at least in broad outline, a genuine renovation of the old theory of invariants, by bringing it into a most intimate relation with some of the main currents of modern algebra. The author's point of view is in fact distinctly more general than any to be found in the textbooks. For whereas these restrict themselves in the main to the invariants of forms, that is, of symmetric tensors, in the case of Weitzenböck including also antisymmetric tensors, Weyl takes the logical step of allowing his invariants to depend on " quantities " which transform according to an *arbitrary* irreducible representation of the initial group.

We must observe, however, that the author has deliberately restricted himself to giving the main lines and the big theorems. We get few details. The reader must not expect to find a complete system of the binary quintic, or even the word transvectant in the index. The book is not, and is not intended to be, a substitute for the older textbooks of invariant theory.

Nevertheless it is extraordinarily rich in matter, so much so that it is really quite difficult to give in a short space at all an adequate idea of what it contains. The reformulation of projective invariant theory is only one, though perhaps the most important, of the main themes. The representations and

invariants of the orthogonal groups $O(n)$ come in for nearly as much attention, their treatment running *pari passu* with that of $GL(n)$, and giving rise to interesting special problems, both algebraical and topological. One further infinite system of simple groups remains to complete the Killing-Cartan classification, and at the same time the list of classical groups. These are the groups defined by a non-singular antisymmetric bilinear form as invariant. The unfortunate name of *complex* groups by which they have hitherto been known (the allusion is to linear complexes) is here abandoned in favour of *symplectic*, which seems a happy choice. The symplectic groups $Sp(n)$ are not treated in quite so much detail as the orthogonal groups, to which they are in some ways closely analogous.

In one respect, the groups here considered extend distinctly beyond the limits of what could reasonably be called classical. For the field of scalars—the field from which coefficients and coordinates are to be chosen—is not assumed to be the field of real or complex numbers, but is allowed to be any field of characteristic zero. It is rather surprising how small a difference this makes; though doubtless the author was wise to exclude fields of characteristic a prime.

Enough has probably now been said to indicate the great importance and interest of the subject-matter of this book. Turning to the details: Chapter I begins with the definitions of field, ring and vector space. Then, taking up the ideas of Klein's Erlangen programme, it describes the dual role of the notion of group in geometry: first, as group of automorphisms (that is, of point-transformations which preserve certain specified geometrical relations, characteristic of the particular geometry under consideration); secondly, as group of coordinate-transformations, linking together the "preferred" systems of reference of the basic space. Derived from this basic space, and implicit in it, are other vector spaces of various dimensions, such as the dual space or the tensor spaces. By considering the effect of an automorphism of the basic space on one of these derived spaces, we arrive at the key notions of *induced transformation*, of *representation* (of the basic group of automorphisms) and of *quantities* (that is, vectors subject to a specific law of transformation, and therefore "belonging" to a definite representation of the basic group). Invariants and covariants are simply scalar quantities. These ideas form the essential background of all that follows.

Chapter II deals with vector invariants, that is, invariants of linear forms, both for $GL(n)$ and $O(n)$, and including contravariant arguments. Using the Capelli identities and Cayley's parametrization of the orthogonal group, the first and second main theorems of invariant theory are proved, showing that any vector invariant can be expressed in terms of the standard bracket invariants, and giving a complete set of algebraic relations connecting these.

Chapter III contains a beautiful account of the modern theory of completely reducible matric algebras, or, what is much the same, of the representation theory of semi-simple algebras, with the ring of a finite group as an example. The theory hinges on the relation of reciprocity between such an algebra and its commutator algebra, that is, the algebra of all matrices which commute with every matrix of the given algebra. The use of the word commutator in this sense seems to be open to serious objections; for the word is already widely used in the theory of groups with quite a different meaning, due to Dedekind. The objection is not diminished by the fact that there is a perfectly good word "centralizer" ready to hand for the important notion in question.

Chapter IV contains the analysis of tensors already referred to. And the next two chapters give a similar complete reduction of the tensorial represen-

tations of $O(n)$ and $Sp(n)$ respectively, including also in the former case the proper orthogonal group $O^+(n)$.

Up to this point the arguments have been essentially algebraical. In Chapter VII, which is devoted to the calculation of the characters of the irreducible representations, we are introduced to the idea of the invariant element of volume of a compact group. This allows us to perform integrations over the group manifold, and so provides a substitute for the summations of finite group theory. The author begins by showing that nothing of algebraic import is lost by restricting attention to the unitary sub-groups of the classical groups. Since these unitary sub-groups are compact, the way is open for analytical methods. The results thus obtained are then derived a second time by purely algebraical arguments.

The book concludes—apart from a short Chapter IX rounding off the algebra theory of Chapter III—with the general theory of invariants in Chapter VIII. After an explanation of the symbolic method, we are given an algebraic proof of the first main theorem for $GL(n)$, using the Cayley Ω-process. The result is then extended to the other groups, including the affine group, by an adjunction argument. The author then turns to analytical methods based on Lie algebras ; and, with the help of the integration process, proves the existence of a finite integrity basis for invariants in the case of any compact Lie group. This chapter also contains much of interest on the topology of the classical groups, touching for instance on the topic of spinors.

From the point of view of the abstractionists and axiom-lovers, now so dominant in algebraic research, the matters here described must necessarily appear somewhat special, as the author himself points out. " Nevertheless ", he adds, " I am convinced that the special problems in all their complexity constitute the stock and core of mathematics ; and to master their difficulties requires on the whole the harder labour. . . . The general theories are here shown as springing forth from special problems whose analysis leads to them with almost inevitable necessity as the fitting tools for their solution ; once developed, these theories spread their light over a wide region beyond their limited origin."

The author has woven his particular problems and his general theorems together, into a pattern which is both intricate and beautiful, and this is one of the book's main attractions. The wide range and rich variety of the interest and detail is another. And although the author complains of having to write in a foreign tongue, " not sung at my cradle ", and of the resulting loss in vigour, ease and lucidity of expression, this loss is by no means apparent in the book itself. The highly characteristic style seems as athletic and light-hearted as ever. And if he has not written an altogether easy book, he is clearly incapable of writing a dull one. P. HALL.

An Introduction to Analytical Geometry. I. By A. ROBSON. Pp. xiv, 409. 9s. 1940. (Cambridge)

Mr. Robson has set out to provide a textbook on analytical geometry whose aim is to give the modern approach to the subject and to prepare the pupil for the point of view which will be presented to him later at the university. How far the latter purpose has been achieved will appear with volume II, but in the meanwhile it can safely be said that the present volume is a work of real distinction which forms a sound introduction to the groundwork of analytical geometry. It should be welcomed by school teachers, whether their pupils are to proceed to more advanced work or not.

The most striking feature of the book, characteristic of Mr. Robson's care to encourage more than mere manipulation, is the chapter on " Abstract Geometry ",

in which an attempt is made to justify the use of complex and infinite elements. This chapter deserves the most serious consideration by teachers, and will be discussed in detail later. The other feature which attracts immediate attention is the range of subject-matter. "The aim has been to introduce a large variety of methods and ideas", and the author has not been afraid to tackle any problem that seems relevant, even though it is not usually classed as "analytical geometry *sub-title*, schools".

The first chapter, on coordinates, begins with coordinates on a line and proceeds to two dimensions, giving polars and the standard homogeneous coordinates. Brief mention is also made of the case of three dimensions. In chapter two, on equations and loci, various examples of loci are considered, and parameters are introduced at a very early stage. The illustrations are not confined to conics, but include various algebraic curves, and the reader is taught to observe quickly any symmetry with respect to the axes or the origin.

Chapter three, on the point and line, deals with the usual topics, and concludes with the introduction of line coordinates, using the notation $[X, Y]$ for the line $Xx+Yy+1=0$, followed by a few remarks on duality in two dimensions. Chapter four is devoted to duality and degeneracy; duality is explained by parallel columns, and degeneracy is introduced for loci and envelopes simultaneously, while here again the work is not confined to conics. Chapter five gives the main elementary results for circles, including radical axes and coaxal systems, and there is a short introduction to inversion. This part of the book is rounded off with a set of 105 miscellaneous examples.

Chapter six introduces the reader to parabola, ellipse and hyperbola by means of parametric equations. More general curves are also considered parametrically, and conditions of collinearity and tangency are explained, with applications to the determination of double points. In chapter seven, on the general algebraic curve, the question of tangency is considered in more detail, and the general conic is introduced. The chapter concludes with a brief account of curves of class m.

Chapter eight, on abstract geometry, makes a determined assault on the use of infinite and complex elements. In the preceding chapters, a pair of equations such as $x+y=0$, $x+y=1$ has been regarded as having no solution, while the equation $x^2+y^2=0$ is satisfied only by the point (0, 0), and this point of view has been emphasised continually. But now the geometry is generalised, and a classification is given of seven geometries G_i. The first, G_1, is the geometry of Euclid, and G_2 is the "geometry of graphs due to Descartes". G_3 is an abstraction of these two, being the geometry of ordered pairs (x, y). In G_4, complex cartesian geometry, the pairs are extended to include complex numbers, and in G_5, real homogeneous cartesian geometry, we reach the points at infinity on the line $z=0$. G_6 is the full complex homogeneous cartesian geometry, in which all things are possible; it is important, though, to note that it remains a metrical geometry; for instance, the condition of perpendicularity of two straight lines is $a_1a_2+b_1b_2=0$. The ideal geometry G_7, in which "a point is a triplet of complex numbers (x, y, z) and there is nothing peculiar about the points for which $z=0$", is just mentioned, and will presumably receive fuller treatment in volume II. The "circular points at infinity" are defined, but no details are given in this volume.

Chapter nine treats of conical projection, and leads from it to the general linear transformation. In chapter ten cross-ratio is explained, and the leading properties for ranges and pencils determined. Chapter eleven gives the special case of harmonic ranges, together with an account of the quadrilateral and quadrangle.

Chapters twelve to fifteen deal with the general conic, the parabola, the

ellipse and the hyperbola respectively. Joachimsthal's method is carefully explained, both for locus and envelope, and pole and polar are determined. The particular conics are also discussed in some detail by the methods of pure geometry (a welcome digression) and these four chapters seem to contain all that a boy can reasonably be expected to know at this stage.

The final chapter discusses the conics of the pencil $s = ks'$, with detailed consideration of the usual special cases, giving also the dual forms in equal detail. Concyclic points are also examined, and there is a short note on "equations of more general form". The book concludes with three sets of miscellaneous examples, on conics, curve-tracing, and properties of general curves respectively.

The book shows throughout the evidence of careful preparation, and is written in a concise, graphic style in which every word tells. There are numerous worked examples introducing most of the tricks that one develops in the course of time, and (I found) a few more. The pupil finds himself confronted with over 2,000 examples of all degrees of difficulty, and should be sure of himself when the course is completed. No sources are given for them, and there seems internal evidence that the wording of some of them hasn't been through a panel of examiners!

Inevitably in a book of such originality and scope certain points arise with which not everyone will agree. I was struck in several places with the almost casual way in which points of great importance are introduced; I have in mind such things as coordinates (p. 1), normal (p. 121), or asymptotes (p. 128). The case of coordinates is typical. Although the first chapter is devoted to the subject of coordinates, yet the term is nowhere defined. It is possible at first sight to justify this from the introduction, where the author expects of the reader that "from his study of graphs he will have gained some knowledge of rectangular cartesian coordinates". But the term is used here in a much deeper sense: for instance, we find on p. 21 that "a circle may be represented by three coordinates; these can be the radius of the circle and the cartesian coordinates of its centre". With such extensions impending, some introductory remarks seem necessary.

Even more striking is the failure to make any introductory comments on the reason for avoiding a zero denominator. The very first case arises on p. 2, where it is proved that the point $\left(\frac{\kappa_1 x_1 + \kappa_2 x_2}{\kappa_1 + \kappa_2}\right)$ divides the line joining the points (x_1), (x_2) in the ratio $\kappa_2 : \kappa_1$. "This important formula applies whether κ_2/κ_1 is positive or negative, but $\kappa_1 + \kappa_2$ must not be zero." And that is all. On p. 3, where homogeneous coordinates (κ_1, κ_2) are defined on a line, we are told that "the pair of coordinates 0, 0 corresponds to no point". And that, again, is all. As this whole question is emphasised again and again throughout the first seven chapters, it would certainly seem to deserve a preliminary reference. Incidentally, since Mr. Robson has set himself such a high standard in watching infinities, we must not allow him to imply (p. 69) that *every* pair of perpendicular lines through the origin is included in the form

$$x^2 + 2cxy - y^2 = 0.$$

A fundamental point which has given me trouble is the exact standard of the pupil for whom this book is intended. In the preface it is stated that "a previous course of coordinate geometry is not necessary". But a knowledge of "elementary calculus" is assumed explicitly, and as early as p. 12 we find reference to "the modulus and principal value of the amplitude of the complex number represented by P in the Argand diagram", while determinants are

used on p. 14. I imagine that few boys will have reached the implied stage without some knowledge of elementary analytical geometry.

I raise this point here because the preparation of the whole book is bound up with it. For on the one hand Mr. Robson introduces certain subjects at what seems an unduly early stage, while others are held back which, with the assumption of even a mild acquaintance with the elements, would tidy up the presentation considerably. As an example of the first case, I quote the end of chapter two, where after forty pages of analytical geometry the pupil is introduced to the space curve $x(t)$, $y(t)$, $z(t)$ and the surface $x(u, v)$, $y(u, v)$, $z(u, v)$. While everyone will sympathise with Mr. Robson's aim to give the reader a broad view of the field, I feel that two pages at so early a stage are not likely to be of great service—and may even be confusing at the hands of an unskilful teacher.

The other extreme is represented by the subject-matter of chapter eight, on Abstract Geometry. It is obviously undesirable to introduce the full details at too early a stage, and yet matters keep arising from the second page onwards which call for resolution and have to be kept waiting. To the reader who knows the answer, it is all right; but the beginner is likely to feel confused and somewhat stranded. Two examples will illustrate the difficulty. Consider the condition that the general equation of the second degree should represent two straight lines; on p. 74 the condition $\delta = 0$ in the usual notation is proved necessary in the cartesian case; that it is not sufficient is demonstrated by the quadratic $x^2 + y^2$. It is not until p. 169 that the sufficiency condition is completed. [But all honour to Mr. Robson for making the fundamentals so clear.] As a second illustration, we may refer to one of the examples, p. 60, ex. 17, where the reader finds that lines through the origin (a not inconsiderable class!) have no line coordinates. I believe that Mr. Robson could have "let in" a few pages quite early which would have given the pupil some idea of the solution of his difficulties, while reserving the detailed discussion to the later chapter. Thus, without loss of rigour, the subject might have been presented more concisely in places, and the pupil saved a delay of many pages before his natural questions receive their answer.

The standard of accuracy seems very high, and a boy taught from this book should have little to unlearn when he comes to more advanced work. The stage seems well set for volume II. For the convenience of the reader, one or two slips or misprints are noted. On p. 14, § 1.71, read $\Delta P_1P_2P_3$; on p. 90, the tangent at the point t is $x \cos t + y \sin t = a$, with a similar correction on p. 91; on p. 146, line 12, for "three" read "four"; on p. 151 the cubic c' should read $x(x^2 - 1) - 3y(y^2 - 1)$; and on the same page, line 25, the reference should surely read § 12.9.

I have not been through all the 2,000 or so examples! But as some check of their accuracy is desirable, I have worked a fairly large number of them, including the whole 98 of miscellaneous B. I found no errors (though an "in general" seems wanted at times), but there are one or two misprints. On p. 191, ex. 17, the equation seems wrong: I think $(x^2 - x + 1)^3 = c(x^2 - x)^2$ is intended; on p. 353, ex. 2, the first circle is $(x - a)^2 + y^2 = b^2$; and something which I cannot decipher has gone wrong with either me or p. 355, ex. 25.

Three minor points (all of notation) bothered me. I can see no good reason for writing the homogeneous quadratic form in the order

$$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2;$$

the whole symmetry of the expression is lost, and the schoolboy deprived of a justification for the apparently perverse notation $x^2 + y^2 + 2gx + 2fy + c$ which he is seldom comfortable in accepting. The second point refers to the order of

symmetry for four elements A, B, C, D. Mr. Robson separates out A, and changes B, C, D cyclically. Thus in a quadrangle the diagonal points are $X=(AB, CD)$; $Y=(AC, BD)$; $Z=(AD, BC)$. It is surely better, and more natural, to separate out the new element D, and permute A, B, C. Thus $X=(BC, AD)$; $Y=(CA, BD)$; $Z=(AB, CD)$. A small matter, certainly, but I believe that it leads to a better sense of form. The advantages perhaps show up better when dealing with coordinates x, y, z, t, ready for dealing with the symmetries in a more general space $x_0, x_1, \ldots, x_n$. My last grumble is trivial. The numbering of paragraphs (which bids fair to become a new branch of mathematics) is by a decimal system 1.1, 1.2, etc., with sub-divisions 1.11, 1.12, 1.13, etc., and is very convenient; but suddenly we find, for no reason that I can discover, sequences such as 3.2, 3.25, 3.3 or 5.31, 5.32, 5.35, culminating in the delirious sequence 15.71, 15.72, 15.78, 15.79 (for which, however, I have discovered an explanation!).

I have finished this review by airing some grievances; but they are small compared with the many merits of the book, and it is to be hoped that Mr. Robson's careful exposition will remove a lot of the untidy thinking which is common in elementary geometry. Those who like manipulation will get any amount of it throughout the book, but it is kept in hand (appropriately enough!), and the reader should finish with a real understanding of the subject, ready for the more advanced work which we may hope to find soon in volume II. E. A. MAXWELL.

Elementary Mathematics from an advanced standpoint: Geometry. By FELIX KLEIN: translated by E. R. HEDRICK and C. A. NOBLE. Pp. ix, 214. 15s. 1939. (Macmillan)

This is an addition to the debt which English readers already owe to Professors Hedrick and Noble. Their translation of the earlier volume (Arithmetic Algebra, Analysis) was reviewed in the *Gazette* in 1933 (XVII, 142). The book is based on lectures delivered more than thirty years ago by a mathematician of the first rank, who was actively interested in the reform of school teaching and in the relations between school and university mathematics and the preparation of university students for their careers as secondary school teachers. The whole book, and especially this geometry section, is concerned less with the details of teaching than with the background required by the teacher. It will be of great interest and value to those teachers who have not read the German edition, and perhaps of even greater value to those who are concerned with the preparation of courses in the training colleges. It is not a self-contained book. As Klein himself says: it is a framework into which should be fitted the separate items of knowledge which the reader has already acquired in the course of his study. If the reader is an average teacher, he will also need to supplement his knowledge. References are given which will help him to do so; these are naturally not always the same references that would be given by an English lecturer in 1940. But it is astonishing how much of Klein's advice remains applicable and necessary after so many years. For example, the fusion of mathematical subjects is nowadays generally recommended and so is the method of introducing new functions graphically by means of the area under a curve, but it can hardly be said yet that these recommendations are superfluous.

The first part of the book deals with the content of geometry. No reader can fail to realise the fundamental importance of such subjects as matrices, determinants, duality, and vectors. Cayley once remarked to Klein that if he had to deliver fifteen lectures on the whole of mathematics, he would devote one of them to determinants; it is interesting to speculate what the number

would have been to-day. By a systematic geometrical investigation Klein arrives at just those quantities which the physicist must have at hand in his study of vector-fields. He goes on to remark: "It is pure geometry however that we are studying. . . . In the course of time physics developed all sorts of mathematical needs. Hence it often created valuable stimulation to mathematical science. But mathematical instruction . . . pays no attention to these changes. It goes along in the same old rut . . . and leaves it to physics laboriously to provide its own aids, although these would supply much more appropriate material for mathematical instruction than do the traditional topics."

The second part deals with geometric transformations and suggests that the fundamental ideas and the simpler parts of this subject also provide stimulating material for school instruction.

The third part is concerned with a systematic discussion of geometry and its foundations. Cayley's principle that "projective geometry is all geometry" is quoted and it is shown how it leads to a systematic development of geometry as a whole. There are sections on invariants and their application to geometry. At the end of the discussion on the foundations of geometry there is an interesting account of Euclid's *Elements*, with estimates of its place in history and of its scientific importance.

In 1933 a reviewer suggested that those who were beginning to teach mathematics should be persuaded or compelled to read the earlier volume. If this has been done, it may be expected that they will read the continuation on their own initiative.

Considerable reforms have been carried out in recent years in the teaching of the most elementary mathematics. Changes in sixth-form mathematics have been more gradual; some of them might have been accelerated if the translators had done their work twenty-five years sooner, and some other reforms might have been introduced. A. R.

Essai sur l'Incompréhension mathématique. By R. DUGAS. Pp. iv, 130. 25 fr. 1940. (Vuibert, Paris)

Nowhere is failure to understand more exasperating and more frequent than in the study of mathematics. Here occur none of the peculiar difficulties of communication which vex the propagandist or the artist. The elementary ideas are clear, universally accessible and expressed in language designed to facilitate thought and eliminate irrelevant associations. Or so, at least, learners are assured. Unable to find respectable excuses for their own failure, those who have "no head for mathematics" (say ninety-nine per cent. of the reading public) are forced to believe that their own mathematical incapacity is matched only by the extreme difficulty of the subject.

The hosts of the mathematically illiterate might get some comfort and a little malicious pleasure from M. Dugas' acute diagnosis of the causes of failure to understand mathematics. (Unfortunately due appreciation of his argument would require a wide knowledge of mathematics.) M. Dugas is hardly concerned with individual impediments to understanding. Such personal difficulties as inability to educe relations or to evoke appropriate imagery are subordinate in his analysis to the broad features of mathematical method which tend *in general* to hinder understanding. For this reason he is able to assume even a high degree of mathematical aptitude in those who provide him with examples. Because certain pervasive features of mathematical method provoke misconstruction we are privileged to watch even the pundits stumbling. This essay is therefore in a sense an apology for incompre-

hension, an analysis of its objective causes which might provide the basis both for excuse and for remedy.

We are invited to attend once more to the axiomatic character of pure mathematics, from which springs three main causes of misunderstanding. There is the habit of confining a mathematical term to the more specific and concrete meaning which it has in common affairs, a kind of inelastic conservatism which regards a spade as a spade and a line as something to be drawn with a ruler. Poincaré described this attitude in a striking passage which deserves to be quoted again from M. Dugas' book :

" Nine-tenths of your pupils want weapons for the struggle with the reality of the sensible world . . . they try to back each word by a sensible image, which each level of definition must evoke, transform, develop . . . they do not hear the argument, they watch the figure ; they think they have understood when they have only seen."

Such a disposition M. Dugas labels *l'incompréhension sensualiste*. In moderation such reliance upon sensible intuition is conducive to understanding and discovery. M. Dugas, indeed, is far from holding that pure mathematics consists in drawing inferences from tautological premises. While stressing the foundation in sensible experience of departments of pure mathematics as abstract as the theory of sets of points, he recognises with justice the degree of elaboration to which the experience must be submitted. It is the lack of elasticity, the excessive bondage to what is non-typical in the concrete example, which works mischief. We could regard this failing as a special case of the vice of misplaced concreteness.

A complementary failing, the second in M. Dugas' catalogue, consists in the wholesale sacrifice of intuition, a too complaisant subjection to the formal argument : " accepting with closed eyes the definitions offered, without trying to relate them by analogy to some field of intuition ". So arises *l'incompréhension formaliste*, the vacuity of the student who understands the words without knowing the sense, who can get right answers without knowing why.

The third failing is rarer—a kind of conscientious objection to the murder of common sense in the name of mathematical generality. This is a more healthy form of conservatism, tinged with scepticism, to which the great have been particularly liable. Berkeley storming against " the ghosts of departed quantities ", Descartes' description of negative numbers as *falsae, seu minores quam nihil*, Kronecker's contempt for transfinite number theory are examples of *un refus métaphysique* for which M. Dugas has no difficulty in multiplying illustrations.

M. Dugas is engaged in diagnosis, not in treatment. But his chapter " Du danger de la facilité " and the trend of his scattered pedagogical comments show his sympathies with the historical mode of treating a topic. Attention to the evolution of mathematical ideas, stressing the general outlines of the development but neglecting uninstructive deviations and confusions, he urges, might free the learner from the crippling myth of the perfection of mathematics and encourage self-confidence. The note tracing the history of continuity from Euler to Baire provides a good example of the kind of treatment likely to prove effectual.

The section on mechanics contains some of the shrewdest remarks in the book. The simplified outline of quantum mechanics, for instance, modestly presented as " un exercice pratique d'incompréhension " is more lucid than some pretentious expositions occupying twenty times the space. There are lengthy notes on failure to understand the infinite, continuity (already noticed), some (rather disappointing) " practical exercises " and an appendix

on celebrated errors. The latter is largely based on M. Lecat's book (*Erreurs des mathématiciens*, 1935).

The analysis made by M. Dugas, amply documented as it is, deals with conditions of such generality as to raise occasional doubts of its usefulness. Is it perhaps on the level of a classification of physiological disorders into those (1) originating within the body, (2) produced by intrusion of foreign bodies, (3) occasioned by a combination of (1) and (2)? Such reflections are not quite fair. M. Dugas' broad survey certainly needs supplementation by a meticulously empirical study of the origin and character of errors made by students of mathematics. But a detailed survey of this sort would need a philosophic background, precise enough to suggest concrete hypotheses but sufficiently free from doctrinal prejudice for harmful *a priori* assumptions to be absent.

Such a background M. Dugas' book seems admirably designed to provide. It is tolerant, well-informed, good-humoured; and constructed, like so many French works of a semi-philosophical nature, with an elegance which makes it a pleasure to read. M. BLACK.

Grundlagen der Mathematik. II. By D. HILBERT and P. BERNAYS. Pp. xii, 498. RM. 43.80. 1939. Grundlehren der mathematischen Wissenschaften, 50. (Springer, Berlin)

The first volume of this work was reviewed in the *Gazette* for December, 1934, by the present writer, and the general ideas behind Hilbert's work were there described. It will perhaps be sufficient to say here that the subjects of Hilbert's investigations are demonstrations, supposed to be set out in full detail in the symbolism of mathematical logic; in particular, he discusses arithmetical demonstrations, set out in this way without any word of the ordinary language, and moving forwards by definite rules, which correspond to the laws of logic. He views these demonstrations from outside as a pure play of symbols and discusses the properties of such assemblages of symbols. In this work mathematics becomes, as it were, self-conscious, turning its weapons on itself. The goal which Hilbert had first in mind was a proof of the self-consistency of analysis, a proof that no demonstration as understood above could end in $0 \neq 0$. This aim so far has not been attained and it has been replaced, for the time being, by the humbler aim of showing the self-consistency of arithmetic.

While this programme was still under way, a bombshell was dropped by Gödel in 1930. In a remarkable and famous paper, Gödel showed that it was impossible to prove the self-consistency of arithmetic within the arithmetic scheme itself. For suppose we had before us any demonstration using only the symbols and axioms of logic and arithmetic which terminated in an array of symbols whose interpretation was the assertion of this self-consistency, then from this demonstration another could be constructed whose conclusion, when interpreted, would mean that arithmetic was *not* self-consistent. The proof of the impossibility of proving self-consistency assumes that arithmetic is, in fact, actually self-consistent, and the argument extends to any consistent doctrine which includes arithmetic.

A further result in Gödel's paper is equally important. Given a system of axioms in any doctrine and the laws of logic, all formalised, can we devise a mechanical method which will enable us to discover whether any particular proposition, written in the symbols of the formalism, can or cannot be deduced from the axioms? This is the decision problem. For the logic of simple propositions it is easily solved. But Gödel showed that in arithmetic, or any wider

scheme, it is possible to state theorems which can neither be proved nor disproved within the scheme.

It has sometimes been the occasion of surprise that analysis can prove theorems on integers whose proof in terms of integers only (or what is the same thing, by algebra only), is unknown, and it has been suggested that proofs by integers only must necessarily exist. Whether the latter is the case for specific theorems in the theory of numbers is not yet known, but it is now clear that there must be theorems on integers whose truth or falsehood cannot be decided within the arithmetic field, that is, on the basis of Peano's axioms and definitions by induction. Analysis essentially deals with variable infinite classes of integers, and this widening of the logical field enlarges the class of probable propositions of the narrower field. A similar situation holds in all branches of mathematics which include arithmetic.

Gödel's method is no less important than his results. He replaces by numbers the symbols used in a formalised demonstration; logical combinations of the symbols are replaced by functions of the corresponding numbers. Thus any formula has a number corresponding to it, any demonstration, a series of numbers which can be united by a rule to form a single number. Thus questions on possible demonstrations become questions on a class of numbers.

Other results which must be taken into account are the theorems of Löwenstein-Skolem and Herbrand. The first states that if a system of axioms can be satisfied at all, it can be satisfied by an enumerable model. Thus if the continuum is supposed to be given as a whole, and we then seek to characterise it by axioms, these axioms could also be satisfied by an enumerable set, notwithstanding Cantor's theorem that the continuum, supposed given, is not an enumerable set. In the enumerable model, the set of elements corresponding to the rational numbers and the set corresponding to the irrationals cannot be put into any one-one correspondence which would represent the impossible one-one correspondence between rationals and irrationals. Skolem's theorem seems to indicate that no system of axioms can adequately describe the continuum. Another result of Skolem's shows that no system of axioms of an entirely "constructive" nature can adequately describe the system of integral numbers: we have to use "class" or "property" in a general, non-constructive sense somewhere, say in the axiom of induction.

Herbrand's theorem concerns general demonstrations and is valuable in itself, and also because it leads to a discussion of cases where the decision problem can be solved.

The present volume begins by introducing a peculiar logical axiom, the ϵ-axiom. It should first be stated that all propositional functions, or predicates, as they are called here, are fixed, such as "greater than", and not variable. Hilbert assumes that if a predicate can be satisfied at all, we can pick out an object which satisfies it. This is a mild form of the axiom of choice; it is, however, quite harmless, for it is shown that if a demonstration of a formula free from the ϵ-symbol, (the selection-symbol), is before us, we can in a finite number of steps eliminate the symbol and the use of the corresponding axiom from the whole demonstration. The axiom and symbol are thus mere auxiliaries.

This is shown by means of an earlier theorem which states that when the final formula of a demonstration contains no bound variables, (no all-sign or existence sign), then all the bound variables can be eliminated from the demonstration.

These theorems are applied to give a proof of Herbrand's theorem and of the Löwenstein-Skolem theorem so far as it concerns the predicate calculus, and also of Gödel's completeness theorem that any non-contradictory formula

in the predicate calculus can be satisfied in the domain of integers—assuming this domain is self-consistent.

Next follows a very important chapter which gives a full account of Gödel's method and results, and exhaustively treats certain points not fully proved in Gödel's paper.

We have said that the self-consistency of arithmetic cannot be shown inside arithmetic: a wider scheme is needed. Since the publication of the first volume, Gentzen has succeeded in proving the self-consistency of arithmetic by using the lower ranges of transfinite ordinals (those of the second class). To each demonstration in arithmetic he ascribes such an ordinal, and his argument essentially is this: that if some demonstration were before us which ended with $0 \neq 0$, a demonstration of lower ordinal could be constructed from it, with the same final formula. No proof of this is given in the book, but observations are made on the method used which transcends the finite proof-methods of the book.

The main part of the book ends here. But this is not all: there are four long appendices. The first gives a résumé of the logical and mathematical foundations treated fully in the first volume. The second discusses computable functions—these correspond in Gödel's method to demonstrable propositions—and Church's proof that the full predicate calculus does not admit a solution of the decision-problem. If this problem could be solved, we should have a mechanical method for ascertaining the truth or falsehood of Fermat's last theorem. The third deals with certain sub-sets of the fundamental logical axioms, and is of interest in connection with positive logics, intuitional logics, and so on. The last appendix of nearly fifty pages discusses in concise fashion the foundations of analysis.

The book is thus a torso; the main programme is just unfinished, since Gentzen's theorem is not proved, and a plan of campaign for further work is indicated. But let us be profoundly thankful that publication was not further, and perhaps fatally, delayed in the interests of completeness, for we now have a detailed account of many of the thoughts of the great German master on the foundations of that science which he has enriched by so many important discoveries. H. G. F.

Generalized Foreign Politics. (British Journal of Psychology, monograph supplements, No. XXIII.) By L. F. RICHARDSON. Pp. viii, 91. 8s. 6d. 1939. (Cambridge University Press)

This is an attempt to apply mathematics to the growth of armaments and to the interactions between nations. The reviewer must confess that he began reading it with a strong prejudice against the possibility of such applications. A disturbing factor, a Napoleon or a Lenin, may so greatly influence events. While this is true, their efforts, after all, are limited by circumstances, and it is quite legitimate to investigate what might be called, in comparison, natural changes.

The author begins with the case of two nations, and supposes that, if x, y be their annual expenditures on armaments, then

$$dx/dt = ky - \alpha x + g,$$

where k is a defence constant, α a fatigue constant, g a constant representing fixed grievances. Thus the rate of increase of armaments of one nation is assumed to be proportional to the threats from the other, apart from the fact that each new step in increasing armaments is more difficult than the previous steps, and that a nation with grievances will increase its armaments anyway.

As soon as we have linear differential equations we are on a well-trod path and may expect either periodic phenomena, as in Volterra's work on the

interaction of species, or secular trends. Here we have secular trends either to stability or instability. The assumption that the equations are linear may of course, be entirely unjustified, but a check is provided by the arms race of 1909-14 between the two competing groups. The agreement with the theory is remarkably close, and if the question were one of physics, it would be regarded as settled. But in physics we believe that laws hold, in politics we may not unreasonably doubt it, and many more confirmations would be needed to produce conviction.

Next is discussed trade-cooperation as an offset to military competition, and then the theory of the interaction between several nations. It is shown that if each pair drawn from three nations is separately unstable, they form an unstable triad, and may do so even if each pair is separately stable ; that an increase in communications, such as the invention of aviation, increases tendencies toward instability, and that under certain not unreasonable assumptions, equality between nations makes for instability. The latter result may cause surprise. But has the balance of power ever prevented wars?

The paper was published in June, 1939, and numerical application of the theory showed that the world was in an unstable state. This deduction has since been confirmed.

The author hopes that a further understanding of the causes of war may result in its elimination. Unfortunately, if this theory is developed, it will only be an added piece of knowledge, like the discovery of a new explosive, and it might be used for good or for evil. Nevertheless here we have the first attempt to treat the international problem exactly and it is to be hoped that it will receive the widest possible attention. H. G. F.

Algebra. By G. W. BREWSTER. Pp. viii, 391. 5s. 1939. (Oundle School Bookshop)

This is an interesting book and perhaps a forerunner of the type of book we shall all be wanting if such recommendations as those of the Spens Report are carried into effect.

The first 200 pages cover the usual pre-certificate course in a way that should fully satisfy those who like a rather short book without a great variety of alternative examples. It may be noticed that graphs come perhaps a little later than is now usual and include solution of quadratic and cubic equations before quadratic equations have been solved algebraically. The various chapters are, however, so arranged that they can be taken in different orders at the teacher's discretion.

Work on formulae is stressed and forms an attractive feature. Is the word " reversal " for " change of subject " a novelty?

The main interest of the book, however, will be found in the last 150 pages, in which a large number of parts of more advanced algebra are dealt with briefly and often in a fresh way which is decidedly taking. For example, as soon as infinite sequences are discussed the ideas of an upper and lower bound are introduced and the rule " a bounded monotonic sequence must converge " stated and explained though without formal proof.

The numerical solution of equations and the calculation of logarithms have more space than is usual for the excellent reason that " boys seem to find them interesting ". A familiar chapter becomes " Arrangement and selection ". Curve tracing includes Newton's diagram.

Logarithmic and exponential series, recurring series, the method of differences, determinants, convergence of series and in fact all the parts of algebra usually done by a first-year mathematical specialist are given a short but pleasing treatment.

The last chapter deals briefly with " some arithmetical theories as a rule regarded as belonging to algebra ", scales of notation, continued fractions and indeterminate equations.

Altogether this is an attractive and thought-provoking book. C. O. TUCKEY.

Arithmetic is Easy. By M. M. ROGERS. Pp. 100. 3s. 6d. 1940. (Evans Bros.)

This book admirably fulfils its avowed purpose of persuading young teachers that arithmetic is easy. It provides a corrective to the view that mathematical work is beyond the ability and outside the interests of the majority of ordinary people, and it gives a valuable restatement of basic essentials, with wise emphasis on pupils' enjoyment of properly graded work and on the importance of diagnostic testing, real-life problems, and self-checking exercises. While, however, the writer shows humour, understanding of children, and familiarity with classroom conditions, there is a surprising absence of acknowledgment of recent American contributions to the improvement of arithmetic teaching. The book, while comforting to teachers in the early stages of their professional life, makes no provision for professional growth by suggestions for further utilisation of the American reports of the researches that lie behind the changes recommended in the text. C. M. F.

Modern Science : A Study of Physical Science in the World To-day. By HYMAN LEVY. Pp. x, 736. 21s. 1939. (Hamish Hamilton)

The first comment on this large volume is that it deserves, and needs, a better binding. Anyone who puts it into a school library (as indeed anyone might) should expect to rebind it within a year. And we may wonder whether this is the fault of the publisher (who perhaps thinks that a guinea is quite enough) or of the author, who seeks to give, in one volume, a survey of the advancing front of science and an exposition of the place of science in human life.

Size is only one of the problems which face the author who attempts such a herculean task. The others mostly centre round the identity of the reader. Who is this " ordinary intelligent person "? Can he solve quadratics? Does he know what a vector is? How far does he understand the workings of a steam engine? Does his curiosity need to be stimulated? Can he spare a guinea for a book on modern science?

Professor Levy's " intelligent person " is certainly not the " man in the street ". At times one suspects him of being Macaulay's schoolboy grown up. And it must be said at once that there are many intelligent citizens who would find the book very hard reading. But teachers of mathematics and any others who have some knowledge of science, especially mechanics, will find it interesting and stimulating and broadening to their outlook.

The real achievement of the book is to show by means of a wealth of interesting and novel examples that the ideas of modern physics are by no means far removed from quite ordinary everyday happenings, that they are in fact a development from what we all know and understand. Science is shown as part of our social environment, both in times past and to-day. Science, moreover, has a unity in its purpose, namely the control of nature by man, though the unity of its development is often lost sight of, owing to its division, for convenience, into " branches ".

The book is so comprehensive that it is impossible to give a fair summary of its contents. Nor do the chapter headings afford much clue, for though the author complains that science has too often been presented in mysterious garb, he dresses up his chapters with such fanciful headings as " Parents and their offspring " (meaning " sequences of numbers "), " The fight for mathe-

matical freedom ", " The marriage contract and the contracting parties—particle and field ".

Part I deals with the social background of science and is largely historical and political. It is not necessary to agree with the author's own political views, which are evident enough without being obtrusive, to say that he is right in insisting that science must be regarded as a factor in the political scene and a major cause of social change.

Part II (entitled " Content and Form ") is largely concerned with energy and transformations of energy, energy-levels and the second law of thermodynamics. It contains good and interesting illustrations and breaks the ground for some of the more difficult ideas of modern physics. Like some other writers on " popular " science, Professor Levy speaks of a " quantum " as a unit of energy (p. 197). It may be difficult to tell the layman just what a quantum is, but that is no reason for telling him something that is untrue. (On p. 569, the value of Planck's quantum of action is given in ergs per sec., while on p. 417, " action " is defined as $\int v\,ds$, so the thoughtful reader may well wonder what the dimensions of this curious quantity really are.) There is also a delightful picture of " Maxwell's demon ", the explanation of which in the text is spoilt by an unfortunate confusion of the words " fast " and " slow ". But these are only minor errors in a most informative and stimulating section.

In Part III he deals with " The Universe of Symbols ", covering, in just over 100 pages, the development of number, algebra, theory of numbers, summation of series, vectors and the calculus.

Part IV is an account of Newtonian mechanics, leading up to a very interesting section on wind-tunnel experiments. Part V is on the development of geometry. Part VI (" What is the Universe? ") may be described briefly as " stars and atoms ", with relativity and quantum theory taken on the way. This leads to Part VII, on radiation, in which there is an interesting account of some illumination problems, to show how we meet modern physics in everyday life. In a short " Epilogue ", the author contrasts the modern " dialectical " method of approaching scientific problems with the older " mechanistic " method.

The book as a whole is a remarkable achievement of synthesis, and though it might repay careful pruning, the task would certainly be a very difficult one. E. H. L.

An Introduction to Economics. By R. C. Lyness and E. R. Emmet. Pp. viii, 372. 4s. 6d. 1939. (Bell)

The authors state in the preface that their book is designed primarily for the sixth forms of schools. They have successfully combined a useful introduction for students who are later to take a university course with a bird's-eye view of the subject for science specialists and those going into business. After some excellent chapters on elementary theory, there are descriptive pages on monetary and industrial organisation, followed by a concise summary of the essential features of, *inter alia*, international trade, public finance and taxation, unemployment and the trade cycle.

The chapters on theory may be criticised for not stating the simplifying assumptions ruthlessly enough. It is salutary for beginners to realise that they are being introduced to an apparatus of thought, illustrated by the simplest of examples which isolate certain factors in order to demonstrate the operation of given principles. The mathematical student may observe that the marginal analysis of price equilibrium is closely analogous to the examination of the stationary values of an energy function to determine

equilibrium in statics. He must not blame the authors for the absence of a complementary section on "dynamics", for economic theory can never become an exact science in the mathematical sense.

The book contains a chapter on "Socialism versus Capitalism" from the point of view of economic advantage which seems inadvisable. The theoretical chapters preceding are necessarily elementary, and imperfect competition, though introduced, is not developed to the extent which modern conditions require for practical application. Secondly, the social aspect—the economics of welfare—has barely been touched upon in the main body of the book describing the working of the capitalist system. The authors are, however, aware of the limitations imposed by these defects.

It remains to add that the book is well indexed, contains some useful numerical examples with answers, and is furnished with a bibliography which is comprehensive without being so vast as to deter the student from further reading. C. H. SHARPSTON.

A Text-book of Heat. II. By H. S. ALLEN and R. S. MAXWELL. Pp. ix, 531-848, xi. 10s. 6d. 1939. (Macmillan)

This excellent volume contains a precise mathematical treatment of the more advanced aspects of the theory of Heat, comprising the laws of thermodynamics, convection, conduction, radiation and an introductory discussion of statistical methods and the quantum theory.

In spite of its predominantly theoretical character, many famous experiments are described in their correct historical sequence, thereby providing the reader with considerable evidence in support of the various fundamental laws and hypotheses from which the present-day theories have been evolved.

The first eleven chapters deal with the laws of Thermodynamics and their application to heat engines and chemical processes. In this connection it is gratifying to observe that the authors have devoted considerable attention to notation and have succeeded in producing one, which is simple, concise and unambiguous, in this respect differing widely from those usually employed in some of the more popular works on Thermodynamics. To quote one instance, the perfect differential is denoted by a clarendon type **d**, to distinguish it from the ordinary increment of an entity which is not a "function of the state". This ingenious device removes a difficulty which experience shows is one of the principal sources of confusion to those desirous of acquainting themselves with the mathematical technique of "heat engines". By this means also the hitherto somewhat nebulous concept of "entropy" is brought well within the grasp of the student.

Five more chapters are devoted to a discussion of convection, conduction and radiation, and include a description of the experiments of Lummer and Pringsheim in verifying the laws of Stefan and Wien.

In conclusion, the work describes the latest statistical results of the Bose-Einstein and Fermi-Dirac schools, tracing the development of the allied theory from its "classical" form to that of the present day. J. H. P.

Statistical Dictionary of Terms and Symbols. By A. K. KURTZ and H. A. EDGERTON. Pp. xiii, 191. 12s. net. 1939. (John Wiley and Sons, New York: Chapman and Hall, Ltd., London)

In 1932 two writers, one a worker in the field of educational statistics and the other in actuarial work, published in U.S.A. a compilation of statistical formulae. This book has now been supplemented by the volume under review. One of the present compilers—the vital statistician—is the same as before: the other is, we believe, another educational statistician. The two writers

obtained the cooperation of an Advisory Council of some thirty leading American statisticians in the six fields of (1) biological and zoological statistics, (2) business statistics, (3) economic and social statistics, (4) educational statistics, (5) mathematical statistics, and (6) psychological statistics. One would therefore hope that in these provinces the dictionary would be accurate and comprehensive. A mathematician will, however, feel uneasy when he notes that, after defining " limit ", the volume states the property of a tangent as " meeting a curve in two or more consecutive points " and defines an angle of one degree as " one three-hundred-sixtieth part of the circumference of a circle ". It is not English usage to define the index of a logarithm as the same as the characteristic, and we do not call numerical integration mechanical quadrature. There is confusion on the question of whether ratio (which is not in this country " also called proportion ") deals with pure numbers, quantities measured in the same units, or measures of unrelated units. We have noticed no misprints in the whole of the book, but there are a certain number of cases that perhaps may be forgiven in view of the purpose of the book of vagueness of terminology in the use of, *e.g.*, " adjacent ", " near ", incompleteness of definition, *e.g.*, for " Order ", and absence of full necessary conditions, *e.g.*, in failing to point out that the definition of the binomial theorem as given is only true for n integral.

Statistics is a rapidly growing study and in consequence omissions may be expected. On the vital statistics side, however, it is strange to find no reference at all to the ideas of net or gross reproductive rates, of fertility and fecundity rates, nor to the alternative Registrar-General's method of compiling standardised death-rates. There are a number of names here—with some serious omissions—but no dates nor formulae. Of English statisticians there are, *e.g.*, Pearson, Sheppard, Yule, and Spearman. It is generous to ascribe the product moment method to K. P. though he refused to accept this ascription, whilst what the dictionary here calls the Pearson's cosine π method of correlation is what Pearson himself said was due to Sheppard. The other modern great English school is not referred to at all although the Advisory Council included a number of workers who have worked with R. A. Fisher. His z is here, and so is " likelihood " and " replication ", but his name, and entries under fiducial probability, probit, Latin Square, confounding, interaction, etc., are all missing, in spite of several references to various kinds of sampling.

We can, however, commend the book as a handy and usually accurate compilation of some 2,000 terms used in various ways in modern statistical work. It is well printed, clean, of handy size, and in a serviceable hard-wearing flexible binding. At 12s. it is dear for English pockets. FRANK SANDON.

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THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be below the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. The names of those sending the questions will not be published.

The Secretary would be glad to receive any solutions that have not yet been returned.

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